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ROBUSTNESS RESULTS IN LQG BASED  
MULTIVARIABLE CONTROL DESIGNS

by

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## ABSTRACT

The robustness of control systems with respect to model uncertainty is considered using simple frequency domain criteria. Both old and new results are derived under a common framework in which the minimum singular value of the return difference transfer matrix is the key quantity. In particular the LQ and LQG robustness results are discussed.

## I. INTRODUCTION

The importance of obtaining robustly stable feedback control systems has long been recognized by designers. Indeed, a principal reason for using feedback rather than open loop control is the presence of model uncertainties. Any model is at best an approximation of reality, and the relatively low order, linear, time-invariant models most often used for controller synthesis are bound to be rather crude approximations.

More specifically, a given system model can usually be characterized as follows. There is a certain range of inputs typically bounded in amplitude and in rate of change for which the model gives a reasonable approximation to the system. Outside of this range, due to neglected nonlinearities and dynamic effects, the model and system may behave in grossly different ways. Unfortunately, this range of permissible inputs is rarely spelled out explicitly along with the model, but is rather implicit in the technology that the model came from - there is no "truth in modeling" law in systems theory.

In classical frequency domain techniques for single-input, single-output (SISO) control system design, the robustness issue is naturally handled.<sup>1</sup> These techniques employ various graphical means (e.g., Bode, Nyquist, inverse-Nyquist, Nichols plots) of displaying the system model in terms of its frequency response. From these plots, it is automatic to determine by inspection the minimum change in the model frequency response that leads to instability.

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<sup>1</sup>See the fundamental work of Bode [6], and any good classical textbook, but especially [9].

Commonly used measures of the closeness of a SISO feedback system to instability are its gain and phase margins. The margins are defined with reference to Fig. 1.1. Here the nominal feedback system (with  $\alpha \equiv 1$ ) is assumed stable. The positive phase margin is the smallest value of  $\phi$  greater than zero such that the system of Fig. 1.1 with  $\alpha(j\omega) = e^{j\phi}$  is unstable. The negative phase margin is defined in an analogous fashion. The upward gain margin is the smallest value of  $\alpha = \text{constant} > 1$  for which the system is unstable (usually expressed in decibels with respect to  $\alpha = +1$ ), and the downward gain margin is similarly defined. The notions of gain and phase margins have gained such widespread acceptance that they are commonly incorporated into the specifications for a control system design.

To give a concrete and explicit example, consider the military specifications on the design of flight control systems for piloted aircraft. Among other requirements, a feedback control systems must have certain gain and phase margins in order to be acceptable. To quote from the military specifications

"Stability margins are required for FCS to allow for variations in system dynamics. Three basic types of variations exist:

- Math modeling and data errors in defining the nominal system and plant.
- Variations in dynamic characteristics caused by changes in environmental conditions, manufacturing tolerances, aging, wear, noncritical material failures, and off-nominal power supplies.

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<sup>1</sup>See the fundamental work of Bode [6], and any good classical textbook, but especially [9].

Maintenance induced errors in calibration, installation and adjustment."

It would seem from the above quotation that the robustness issue is well-understood from the viewpoint of classical frequency domain techniques, at least for flight control systems. Indeed, this is the case, for single-loop systems. However, the situation is quite different for multiple-loop systems. To quote again from the military specifications:

"In multiple-loop systems, variations shall be made with all gain and phase values in the feedback paths held at nominal values except for the path under investigation."

The fundamental difficulty with this approach is that it fails to check the affect of simultaneous gain and phase variations in several paths. Of course, real-world model uncertainty cannot be expected to nicely confine itself to a single loop of the system! In fact, for a flight control system, the dominant variation is due to the change in control surface effectiveness with dynamic pressure, which manifests itself as a change in the gains of the transfer functions from control surface deflections to the response variables of interest. The dynamic pressure variation is due to changes in aircraft altitude and speed, and clearly affects all loops simultaneously.

From this discussion, it is clear that a satisfactory notion of stability margins for a multivariable feedback system must be able to characterize the ability of the system to tolerate gain and phase variations in all its loops simultaneously. It is only very recently, in the context of studying the feedback properties of controllers derived using linear-quadratic Gaussian (LQG) techniques, that an appropriate formulation has emerged. The purpose of this paper is to develop the ideas of this formulation, and

to derive in a relatively simple way the robustness properties of LQG controllers.

The paper begins in Section II by developing a characterization of the robustness of a multivariable feedback system in terms of the minimum singular value of its return difference transfer function matrix. This quantity is a natural one to consider as it directly generalizes the classical notion of the distance of the Nyquist locus of a single-input feedback system to the critical (-1) point. The characterization is original, and is derived rather simply from arguments based on the multivariable Nyquist theorem. Multivariable gain, phase, and crossfeed margins are derived in terms of the minimum singular value.

The development in Section II permits an efficient derivation of the robustness properties of LQ controllers in Section II since the Kalman frequency domain inequality provides a bound on the minimum singular value of an optimal return difference transfer function matrix. The dependence of these robustness properties on the form of the control weighting matrix  $R$  is illustrated by introducing a counterexample of an LQ (not LQG) regulator with vanishingly small gain margins. The robustness degradation associated with introducing a Kalman filter into a feedback system are briefly discussed, and robustness recovery procedures are mentioned.

The paper closes in Section IV with a summary and discussion of the results of the paper. Also, since singular values are used throughout the paper, a brief discussion of them is given in the Appendix .

#### Notation

$(A, B, C)$	realization of linear system specified by $\dot{x} = Ax + Bu$ $y = Cx$
$G(s)$	loop-transfer matrix
$\phi_{OL}(s)$	open-loop characteristic polynomial
$\phi_{CL}(s)$	closed-loop characteristic polynomial
$L(s)$	multiplicative perturbation of $G(s)$
$\lambda(A)$	an eigenvalue of $A$
$A^H$	complex conjugate transpose of $A$
$\bar{\sigma}(A)$	maximum singular value of $A = \lambda_{\max}^{1/2}(A^H A)$
$\underline{\sigma}(A)$	minimum singular value of $A \triangleq \lambda_{\min}^{1/2}(A^H A)$
$N(\Omega, f(s), \mathcal{C})$	number of clockwise encirclements of the point $\Omega$ by the locus of $f(s)$ as $s$ traverses the closed contour $\mathcal{C}$ in the complex plane in a clockwise sense.
$D_R$	Nyquist contour of radius $R$ given in Fig. 2.2 with $\frac{1}{R}$ -radius indentations.
$\Omega_R$	segment of $D_R$ for which $\text{Re}[s] \leq 0$ .
SISO	single-input-single-output
MIMO	multiple-input-multiple-output
ORHP(CRHP)	open (closed) right half plane
OLHP(CLHP)	open (closed) left half plane
$A > B$	$A - B$ is positive definite
$A \geq B$	$A - B$ is positive semi-definite

$\iff$  if and only if

$\exists$  there exists

$\exists$  such that

$\Rightarrow$  implies

$\triangleq$  defined as

LQ linear-quadratic

LQG linear-quadratic-gaussian

KF Kalman filter

$z^*$  complex conjugate of  $z$

$\forall$  for all

## II. MULTIVARIABLE STABILITY MARGINS

In this section we will develop the basic characterizations of the robustness of linear multivariable feedback systems, i.e., multivariable stability margins, to be used in the remainder of the paper. The feedback system to be discussed is depicted in Fig. 2.1, where the loop transfer matrix  $G(s)$  is assumed to incorporate both the plant dynamics and any compensation employed, and has the state space realization

$$G(s) = C(sI - A)^{-1} B. \quad (2.1)$$

The basic issue of concern is to characterize the robustness of the feedback system, i.e., the extent to which the elements of the loop transfer function matrix  $G(s)$  can vary from their nominal design values without compromising the stability of the system. The analysis is based on the multivariable Nyquist theorem.

### A. Multivariable Nyquist Theorem

The multivariable Nyquist theorem is derived from the relationship

$$\det(I + G(s)) = \frac{\phi_{CL}(s)}{\phi_{0L}(s)}, \quad (2.2)$$

where

$$\phi_{0L}(s) = \det(sI - A) \quad (2.3)$$

$$\phi_{CL}(s) = \det(sI - A + BC) \quad (2.4)$$

and from the Principle of the Argument of complex variable theory. The following statement of the Nyquist theorem is a variation of a version given by Rosenbrock in [1].

Theorem 2.1: The system of Fig. 2.1 is closed-loop asymptotically stable (in the sense that  $\phi_{CL}(s)$  has no CRHP zeroes) if and only if for all  $R$  sufficiently large

$$\mathcal{N}(0, \det[I+G(s)], D_R) = -P \quad (2.5)$$

or equivalently,

$$\mathcal{N}(-1, -1+\det[I+G(s)], D_R) = -P \quad (2.6)$$

where  $D_R$  is the contour of Fig. 2.2 which encloses all CRHP zeroes of  $\phi_{OL}(s)$ ,  $P$  is the number of CRHP zeroes of  $\phi_{OL}(s)$  and where for convenience, we define  $\mathcal{N}(\Omega, f(s), \mathcal{C}) \triangleq \frac{1}{2}$  when  $f(s_0) = \Omega$ , for some  $s_0 \in \mathcal{C}$ .

Notice that no controllability or observability assumptions have been made. If  $[A, B, C]$  is a nonminimal realization then pole-zero cancellations will occur when  $G(s)$  is formed, eliminating uncontrollable or unobservable modes. The stability of these modes cannot be tested in terms of  $G(s)$ . However, by using the zeroes of  $\phi_{OL}(s)$  instead of the poles of the loop transfer matrix  $G(s)$ , this version of the Nyquist theorem allows one to test for the internal stability of the closed-loop system.

When compared with the classical Nyquist theorem for the SISO case, the multivariable Nyquist theorem is much more difficult to use, for two reasons. First, the dependence of  $\det(I+G(s))$  on the compensation implicit in  $G(s)$  is complicated, and cannot be easily depicted with a Nyquist, Bode or related plot. This fact has motivated a considerable amount of research

on synthesis methods, e.g., [2] - [5], which need not concern us here.

Second, and this is the key observation, one cannot get a satisfactory notion of multivariable stability margins directly from the the multivariable Nyquist theorem. The following extremely simple example illustrates this fact.

Example 2.1:

Consider the linear system specified by

$$\begin{bmatrix} \dot{x}_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & b_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (2.7)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (2.8)$$

which is illustrated in Fig. 2.3.

If the feedback compensation

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} u_{C1} \\ u_{C2} \end{bmatrix} \quad (2.9)$$

is used the closed-loop system is given by

$$\begin{bmatrix} \dot{x}_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & -b_{12} \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & b_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{C1} \\ u_{C2} \end{bmatrix} . \quad (2.10)$$

The eigenvalues of this system are -2, -2 and therefore it is stable. The return difference matrix  $I+G(s)$  is given by

$$I+G(s) = \begin{bmatrix} \frac{s+2}{s+1} & \frac{b_{12}}{s+1} \\ 0 & \frac{s+2}{s+1} \end{bmatrix} \quad (2.11)$$

and thus

$$\det(I+G(s))-1 = \frac{2s+3}{(s+1)^2} \quad . \quad (2.12)$$

The multivariable Nyquist diagram is just the usual Nyquist diagram of  $\frac{2s+3}{(s+1)^2}$

and is shown in Fig. 2.4. If this is interpreted as for a SISO system, one would conclude that the system has an infinite upward gain margin, a gain reduction margin of  $-1/3$  and a phase margin of  $\pm 106^\circ$ -margins indicative of a highly robust system. Note, however, that the Nyquist diagram does not depend on the value of  $b_{12}$  and as  $b_{12}$  becomes large the closed-loop system is close to instability in the following sense. If the open-loop system of Fig. 2.3 is perturbed slightly to obtain the system of Fig. 2.5, the closed-loop system obtained by negative identity feedback is unstable (as is easily verified). This situation cannot be detected by inspection of the multivariable Nyquist theorem.<sup>1</sup>

The difficulty we have uncovered can be explained as follows. A multivariable system will not be robust to modeling errors if its return difference transfer function matrix  $I+G(j\omega)$  is nearly singular at some frequency  $\omega_c$ , since then a small change in  $G(j\omega_c)$  will make  $I+G(j\omega_c)$  exactly singular. When this happens,  $\det(I+G(j\omega_c)) = 0$  and the number of encirclements counted in the multivariable Nyquist theorem changes. Unfortunately, the near singularity

<sup>1</sup>The situation cannot be detected by classical, single-loop tests or by characteristic loci plots [5] either.

of a matrix cannot always be detected in terms of its determinant. Instead, tests such as those developed in the remainder of this section must be employed.

### B. Robustness Characterization

From the above example, we can see that the problem of determining characterizations of the robustness of a multivariable feedback system, i.e., its distance from instability, is of fundamental importance. The basic work in this area is due to Safonov [7], [8], who generalized an approach of Zames [10], [11]. Safonov's work heavily utilizes concepts of functional analysis, as is standard in the modern input-output formulation of stability theory<sup>1</sup>. However, in the finite dimensional linear-time-invariant case considered in this paper, powerful robustness characterization can be derived more simply in terms of the multivariable Nyquist theorem.

The feedback system under consideration in the following theorems is shown in Fig. 2.6, where  $G(s)$  again represents the loop-transfer matrix (open-loop plant plus controller) as in Fig. 2.1 and  $L(s)$  is a multiplicative perturbation applied to  $G(s)$  to account for uncertainty in the open-loop plant model.

We denote the perturbed system as  $\tilde{G}(s)$  given by

$$\tilde{G}(s) = G(s)L(s) \quad (2.13)$$

which is assumed to have a state-space realization  $(\tilde{A}, \tilde{B}, \tilde{C})$ , open-loop characteristic polynomial  $\tilde{\phi}_{0L}(s)$  given by

$$\tilde{\phi}_{0L}(s) = \det(sI - \tilde{A}) \quad (2.14)$$

and similarly a closed-loop characteristic polynomial  $\tilde{\phi}_{CL}(s)$  given by

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<sup>1</sup>See, e.g., [12] or [13]

$$\tilde{\phi}_{CL}(s) = \det(sI - \tilde{A} + \tilde{B}\tilde{C}). \quad (2.15)$$

All the results of this section are based on the following theorem.

Theorem 2.2: The polynomial  $\tilde{\phi}_{CL}(s)$  of (2.14) has no CRHP zeroes if the following conditions hold:

(a)  $\phi_{0L}(s)$  and  $\tilde{\phi}_{0L}(s)$  have the same number of CRHP zeroes, (2.16)

(b) if  $\tilde{\phi}_{0L}(j\omega_0) = 0$  then  $\phi_{0L}(j\omega_0) = 0$ , (2.17)

(c)  $\phi_{CL}(s)$  has no CRHP zeroes, (2.18)

(d)  $\det[I + (I - \epsilon)G(s) + \epsilon \tilde{G}(s)] \neq 0$ ,  $s \in D_R$  for all  $R$  sufficiently large and for all  $\epsilon$  on the interval  $[0, 1]$ . (2.19)

Proof: Appendix

The basic idea behind this theorem is that of continuously deforming the Nyquist diagram of the nominal system  $G(s)$  to one corresponding to the Nyquist diagram of the perturbed system  $\tilde{G}(s)$  without changing the number of encirclements of the critical point. If this can be done and the number of encirclements of the critical point required for  $\tilde{G}(s)$  and  $G(s)$  are the same, then no CRHP zeroes of  $\phi_{CL}(s)$  will result from this perturbation. Imbedding arguments of this type have been previously used, implicitly by Rosenbrock [1] and explicitly by Doyle [14], in connection with linear systems and in the more general context of nonlinear or multidimensional systems by DeCarlo, Saeks

and Murray [15] - [17], utilizing homotopy theory from algebraic topology.

The significance of Theorem 2.2 is that various multivariable robustness characterizations can be stated in terms of conditions that guarantee that (2.19) is satisfied. In checking condition (2.19), it is unnecessary to consider all  $s \in D_R$  because  $\bar{\sigma}[G(s)]$  and  $\bar{\sigma}[\tilde{G}(s)] \rightarrow 0$  as  $|s| \rightarrow \infty$ . This is due to the assumption that their respective state-space realizations have no direct feedthrough from input to output so that there are more poles than zeroes in both  $\tilde{G}(s)$  and  $G(s)$ . It is therefore convenient to define  $\Omega_R$  as

$$\Omega_R \stackrel{\Delta}{=} \{s \mid s \in D_R \text{ and } \operatorname{Re}(s) \leq 0\} \quad (2.20)$$

which is the only part of the Nyquist contour on which (2.19) need be verified. Using this simplification, one characterization that guarantees (2.19) is based on the return difference matrix  $I+G(s)$ . This result is new, although related to conicity conditions discussed by Zames [10] and Safonov [7], and is fundamental to the derivation of the LQ state-feedback and LQG stability margins.

We emphasize that the following theorem is distinct from the main results in [14], [18]. These papers work with  $\underline{\sigma}(I+G^{-1}(s))$  (for a multiplicative representation of model uncertainty), which is complementary to the quantity  $\underline{\sigma}(I+G(s))$ , which measures the distance between the Nyquist locus and the critical (-1) point in the SISO case. Moreover, these papers work with  $\bar{\sigma}(L(s)-I)$  (in our notation) rather than  $\bar{\sigma}(L^{-1}(s)-I)$ . As a consequence, it is not possible to derive the gain and phase margin properties of LQ and LQG controllers using the results stated in [14], [18].

Theorem 2.3: The polynomial  $\tilde{\Phi}_{CL}(s)$  has no CRHP zeroes if the following conditions hold for all  $R$  sufficiently large: .

(a) conditions (2.16) - (2.18) of Theorem 2.2 hold (2.21)

$$(b) \bar{\sigma}(L^{-1}(s) - I) < \alpha \stackrel{\Delta}{=} \underline{\sigma}(I + G(s)), \quad s \in \Omega_R \quad (2.22)$$

(c) any one of the following is satisfied at each  $s \in \Omega_R$

$$(i) \quad \alpha \leq 1 \quad (2.23)$$

$$(ii) \quad L^H(s) + L(s) \geq 0 \quad (2.24)$$

$$(iii) \quad 2(\alpha^2 - 1) \underline{\sigma}^2(L(s) - I) > \alpha^2 \bar{\sigma}^2(L(s) + L^H(s) - 2I) \quad (2.25)$$

An analytical proof of Theorem 2.3 is provided in the Appendix. It is, however, instructive to give a simple graphical proof for the SISO case.

By the embedding argument of Theorem 2.2, our nominal feedback system and its perturbed version will have the same number of encirclements of the critical (-1) point if we can continuously deform the nominal into the perturbed Nyquist locus. This will be the case if, for  $0 \leq \epsilon \leq 1$ , we have

$$1 + [\epsilon \mathcal{L}(s) + (1-\epsilon)]g(s) \neq 0, \quad s \in \Omega_R \quad (2.26)$$

or equivalently

$$-\frac{1}{g(s)} \neq \epsilon \mathcal{L}(s) + (1-\epsilon), \quad s \in \Omega_R. \quad (2.27)$$

In the scalar case, the inequality

$$|1+g(s)| \geq \alpha \quad , \quad (2.28)$$

shown in Fig. 2.7 (d), simply says that  $g(s)$  lies outside a circle of radius  $\alpha$  about the  $-1$  point. Consequently, when  $1+g(s) = \alpha$ ,  $-1/g(s)$  will lie on a circle of radius  $\frac{\alpha}{\alpha^2-1}$  (infinite when  $\alpha=1$ ) centered at  $\frac{1}{1-\alpha^2}$  as illustrated in Figs. 2.7 (a), (b) and (c) for various  $\alpha$ . Then, the allowable values of  $\ell(s)$  are those that can be connected to the  $+1$  point by a straight line not intersecting the circle of possible boundary values of  $-\frac{1}{g(s)}$ .

We emphasize that  $\alpha$  varies as a function of  $s \in \Omega_R$ , so that Fig. 2.7 only represents the situation at a single (complex) frequency. Similarly, the condition (2.22) of Theorem 2.3 must be tested for all  $s \in \Omega_R$ . This is most readily accomplished by computing and plotting  $\bar{\sigma}(L^{-1}(s)-1)$  and  $\underline{\sigma}(1+G(s))$ . The situation is roughly analogous to looking at a magnitude Bode plot. Computational techniques are discussed in [19].

To show that condition (2.22), in Theorem 2.3, alone is not enough to guarantee stability of the perturbed closed-loop system of Fig. 2.6 a simple counterexample is given.

Example 2.2: Let

$$g(s) = \frac{9}{s+1} \quad (2.29)$$

$$\phi_{0L}(s) = s+1 \quad (2.30)$$

$$\phi_{CL}(s) = s+10 \quad (2.31)$$

$$\ell(s) = \frac{s-1}{s+1} \quad (2.32)$$

$$\tilde{\phi}_{0L}(s) = (s+1)^2 \quad (2.33)$$

then

$$|\varrho^{-1}(j\omega) - 1| < |1+g(j\omega)| \quad \forall \omega \quad (2.34)$$

but

$$\tilde{\varphi}_{CL}(s) = s^2 + 11s - 8 \quad (2.35)$$

has a CRHP zero.

From Theorem 2.3, it is clear that the quantity  $\underline{\varphi}(1+G(s))$  is a multi-variable stability margin for the feedback system at the complex frequency  $s$ . However, this quantity is unconventional, and possibly somewhat difficult to interpret. Therefore, we will explore further some of the consequences of the theorem.

Note from Figs. 2.7 (a)-(d) that in the SISO case a system satisfying the conditions of Theorem 2.3 will automatically have certain guaranteed minimum gain and phase margins if  $\alpha$  (i.e.  $|1+g(s)|$ ) as a function of  $s \in \Omega_R$  has a constant non-zero lower bound, say  $\alpha_0$ . If  $\alpha_0 = 1$ , then Fig. 2.7(d) shows that  $g(s)$  must have exactly a one-pole roll-off. It is well-known (see, e.g., [20]) that physical systems always exhibit at least a two-pole roll-off which again from Fig. 2.7 (d) indicates that  $\alpha_0 < 1$ .

The case  $\alpha_0 > 1$  is inconsistent with the assumption that the state-space realization of  $g(s)$  does not have a feedthrough term. Thus for  $\alpha_0 \leq 1$  examination of Fig. 2.7(b) or Figs. 2.7(c) and (d) indicates the guaranteed gain and phase margins. These margins generalize to the multivariable case as demonstrated by the following corollary.

Corollary 2.1: If the conditions of Theorem 2.3 hold for all  $R$  sufficiently large and

$$\underline{\sigma}(1+G(s)) \geq \alpha_0, \quad s \in \Omega_R \quad (2.36)$$

for some constant  $\alpha_0 \leq 1$ , then simultaneously in each loop of the feedback system of Fig. 2.1 there is a guaranteed gain margin (denoted GM) given by

$$GM = \frac{1}{1 \pm \alpha_0} \quad (2.37)$$

and also a guaranteed phase margin (denoted PM) given by

$$PM = \pm \cos^{-1} \left[ 1 - \frac{\alpha_0^2}{2} \right] \quad . \quad (2.38)$$

Proof: Appendix

The interpretation of the gain and phase margin quantities specified in Corollary 2.1 require some explanation. First of all, the word "simultaneously" in Corollary 2.1, means that the gains or the phases of all the feedback loops may be changed at the same time within the limits prescribed by (2.37) and (2.38) without destabilizing the closed-loop system. It does not mean, just as it does not in SISO case, that the gains and phases may be changed simultaneously. Secondly, (2.34) is to be interpreted as meaning any gains,  $\gamma_i$ , inserted in the feedback loops of the system of Fig. 2.1 satisfying

$$\frac{1}{1+\alpha_0} < \gamma_i < \frac{1}{1-\alpha_0} \quad (2.39)$$

will not destabilize the closed-loop system. Similarly for (2.35), every loop may have a phase factor  $e^{j\phi_i}$  inserted provided

$$|\phi_i| < \cos^{-1} \left[ 1 - \frac{\alpha_0^2}{2} \right] \quad . \quad (2.40)$$

and the system will remain closed-loop stable.

The ability to consider simultaneous gain or phase variations in all the loops of a multivariable feedback system is physically very appealing. A typical example is a flight control system, in which the effectiveness of all control surfaces varies simultaneously as a function of altitude and airspeed. Another common model uncertainty in flight control applications is a crossfeed arising from a neglected interaxis coupling. Bounds on the ability of a multivariable system to tolerate crossfeed uncertainty are given in the following result.

Corollary 2.2: Provided the conditions of Theorem 2.3 are satisfied for all  $R$  sufficiently large and

$$\underline{\sigma}(I+G(s)) \geq \alpha_0, \quad s \in \Omega_R \quad (2.41)$$

for some constant  $\alpha_0 \leq 1$ , then the feedback system of Fig. 2.1 will tolerate a crossfeed perturbation of the form

$$L(s) = \begin{bmatrix} I_k & X(s) \\ 0 & I_m \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} I_k & 0 \\ X(s) & I_m \end{bmatrix} \quad (2.42)$$

where  $I_k$  is the  $k \times k$  identity and

$$\bar{\sigma}(X(s)) < \alpha_0, \quad s \in \Omega_R \quad (2.43)$$

without becoming unstable, that is  $\tilde{\phi}_{CL}(s)$  will have no CRHP zeroes.

Proof: Appendix

There is one final result that will be of use to us in the next section. This theorem involves the well-known passivity condition [12], [13].

Theorem 2.4: The polynomial  $\tilde{\Phi}_{CL}(s)$  has no CRHP zeroes if the following conditions hold for all  $R$  sufficiently large:

$$(a) \text{ conditions (2.16)} - (2.18) \text{ of Theorem 2.2 hold} \quad (2.44)$$

$$(b) G(s) + G^H(s) \geq 0, \quad s \in \Omega_R \quad (2.45)$$

$$(c) L(s) + L^H(s) > 0, \quad s \in \Omega_R \quad (2.46)$$

Proof: Appendix

Just as for Theorem 2.3, specializing to the SISO case illustrates the types of  $G(s)$  and  $L(s)$  that are required as well as the associated guaranteed minimum margins. Figs. 2.8(a) and (b) depict the constraints (2.45) and (2.46) in the SISO case and show that since  $g(s)$  cannot encircle the "-1" point it must be open-loop stable in order to apply the theorem. It is fairly obvious that the phase of  $g(s)l(s)$  is less than  $180^\circ$  and thus  $g(s)l(s) \neq -1$  is assured. The gain and phase margins are also immediately apparent from Fig. 2.8(a) or (b) and again generalize to the multivariable case given in Corollary 2.3.

Corollary 2.3: If the conditions of Theorem 2.5 hold for all  $R$  sufficiently large then simultaneously in each feedback loop of the system of Fig. 2.1 there is a guaranteed gain margin given by

$$GM = 0, +\infty \quad (2.47)$$

and also a guaranteed phase margin given by

$$PM = \pm 90^\circ \quad (2.48)$$

Proof: Appendix

Corollary 2.4: If the conditions of Theorem 2.4 are satisfied, then the feedback system of Fig. 2.1 will tolerate a crossfeed perturbation of the form given in (2.42) where

$$\bar{\sigma}(X(s)) < 2, \quad s \in \Omega_R \quad (2.49)$$

without becoming unstable, that is  $\tilde{\phi}_{CL}(s)$  will have no CRHP zeroes.

Proof: Appendix

Returning to Example 2.1, the nearness to instability can be easily detected using Theorem 2.3. Fig. 2.9 shows  $\underline{\sigma}(1+G(j\omega))$  as a function of  $\omega$ , where  $b_{12}$  has been selected as 50. At low frequencies  $\underline{\sigma}(1+G(j\omega))$  is very small giving

$$GM = .93, 1.08 \quad (2.50)$$

and

$$PM = \pm 4.1^\circ \quad (2.51)$$

However, we know by inspection that each feedback loop in the system has an actual infinite upward gain margin and a -1 gain reduction margin which indicates that (2.50) and (2.51) are very conservative estimates of gain and phase margins. Nevertheless, they indicate a robustness problem which is

exhibited by the very small crossfeed tolerance

$$\bar{\sigma}(X(s)) < \min_{s \in R} \underline{\sigma}(I+G(s)) \approx 0.071 = -23 \text{dB} . \quad (2.52)$$

This was precisely the type of perturbation given previously in Fig. 2.5 to show nearness to instability. In general, the type of perturbation that the feedback system is most sensitive to is given by

$$L(s) = [I - \underline{\sigma}(I+G(s)) \underline{u}(s) \underline{v}^H(s)]^{-1} \quad (2.53)$$

where  $\underline{u}(s)$  and  $\underline{v}(s)$  are respectively the right and left singular vectors<sup>1</sup> of  $I+G(s)$  corresponding to  $\underline{\sigma}(I+G(s))$ . This is the perturbation which makes  $I+G(s)L(s)$  exactly singular with minimum  $\bar{\sigma}(\tilde{G}(s) - G(s))$ .

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<sup>1</sup>The right and left singular vectors  $u(s)$  and  $v(s)$  are respectively unit length eigenvectors of  $(I+G(s))(I+G(s))^H$  and  $(I+G(s))^H(I+G(s))$  such that  $(I+G(s))v(s) = \underline{\sigma}(I+G(s))\underline{u}(s)$ .

### III. LQ AND LQG STABILITY MARGINS

The subject of qualitative feedback properties of LQ control systems is not a new one. An early and fundamental paper by Kalman [21] detailed properties shared by all LQ regulators in the SISO case. Kalman showed that the return difference transfer function of a SISO LQ state feedback regulator satisfies the inequality

$$|1+g(j\omega)| \geq 1 \quad \forall \omega \quad (3.1)$$

which is both a classical condition for the reduction of sensitivity by feedback (see, e.g., [9]) as well as necessary and sufficient for a (stable) state feedback regulator to be optimal with respect to some quadratic cost index. By inspection of the Nyquist diagram corresponding to (3.1), (Fig. 3.1), it is straightforward to observe [22, pp. 70-76] that a SISO LQ state feedback regulator has a guaranteed infinite upward gain margin, at least a 50% gain reduction margin and also a guaranteed minimum phase margin of  $\pm 60^\circ$ .

Anderson [23] developed a multivariable version of condition (3.1) as a property of LQ state-feedback regulators; a similar generalized condition arises in sensitivity theory<sup>1</sup> (see, e.g., Cruz and Perkins [24]). In the remainder of this paper, we will exploit the multivariable form of (3.1) together with the results of the proceeding Section to establish the stability margin properties of LQ and LQG optimal regulators.

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<sup>1</sup>Sensitivity refers to the variation in system responses due to infinitesimal changes in the nominal system parameters. Robustness refers to the delineation of finite regions of allowable variation in nominal system parameters that preserve stability.

A. Multivariable Kalman Inequality

We will need a precise statement of the multivariable version of condition (3.1) in the sequel, and this is provided by the following theorem. The proof is by straightforward manipulation of the algebraic Riccati equation and is included, for completeness, in the appendix.

For convenience we will assume that in all remaining theorems and corollaries that the Nyquist contour  $D_R$  is chosen with  $R$  sufficiently large so that the theorems of Section II may be applied.

Theorem 3.1: If  $K$  satisfies

$$A^T K + K A + Q - K B R^{-1} B^T K = 0 \quad (3.2)$$

with  $R > 0$  and  $Q \geq 0$  then

$$(I+G(s))^H R (I+G(s)) = R + H(s) \quad (3.3)$$

where

$$G(s) = R^{-1} B^T K (I - sA)^{-1} B \quad (3.4)$$

$$H(s) = [(I - sA)^{-1} B]^H (Q + 2\text{Re}(s)K) [(I - sA)^{-1} B] \quad (3.5)$$

Furthermore, if  $Q > 0$ ,  $B$  has full rank and  $K \geq 0$  then (3.3) implies that

$$(I+G(s))^H R (I+G(s)) > R, \quad s \in D_R \quad (3.6)$$

Alternatively, if  $\det(j\omega I - A) \neq 0 \quad \forall \omega$  and  $K \geq 0$  then (3.3) implies that

$$(I+G(s))^H R (I+G(s)) \geq R, \quad s \in D_R \quad (3.7)$$

Proof: Appendix

It is important to point out that this theorem uses  $G^H(s)$  rather than  $G^T(-s)$  as in [23]. These two quantities are the same when  $s = j\omega$ , but are different when  $\text{Re}(s) \neq 0$ . This is the case when  $s$  is evaluated along the Nyquist  $D_R$  contour and this contour is indented along the imaginary axis. It is necessary to use  $G^H(s)$  in order to apply the theorems of Section II. Note, however, that when  $\det(j\omega I - A) \neq 0$ ,  $\forall \omega$ , that  $\Omega_R$  is just the imaginary axis from  $-jR$  to  $jR$ . In this case (3.7) could be written as

$$(I + G(j\omega))^H R (I + G(j\omega)) \geq R \quad \forall \omega \quad (3.8)$$

which is the previously mentioned multivariable generalization of condition (3.1).

Of course, the Riccati equation (3.2) arises in connection with the LQ regulator problem given by

$$\min_{u(t)} J(u) = \int_0^{\infty} x^T(t) Q x(t) + u^T(t) R u(t) dt \quad (3.9)$$

$$\text{s.t.} \quad \dot{x}(t) = Ax(t) + Bu(t) \quad (3.10)$$

which has the usual solution

$$u(t) = -R^{-1} B^T K x(t) \quad (3.11)$$

with  $K > 0$  satisfying (3.2), provided  $(A, B, Q^{1/2})$  is minimal.

B. Stability Margins of LQ Regulators

We can now employ Theorem 3.1 in conjunction with the results of Section II to establish the robustness properties of multivariable LQ regulators

Recall from Section II that the key quantity for multivariable robustness analysis is the minimum singular value  $\underline{\sigma}(I+G(s))$ , where  $G(s)$  is the loop transfer matrix. Unfortunately, the inequalities (3.6) and (3.7) of Theorem 3.1 do not provide a bound on  $\underline{\sigma}(I+G(s))$ , where  $G(s)$  is the LQ regulator loop transfer matrix defined by (3.4). However, if we define

$$\hat{G}(s) = R^{\frac{1}{2}}G(s)R^{-\frac{1}{2}}$$

then (3.7) (for example) can be rewritten in the form

$$(I+\hat{G}(s))^H(I+\hat{G}(s)) \geq I, \quad s \in D_R. \quad (3.12)$$

Equation (3.12) provides the bound

$$\underline{\sigma}(I+\hat{G}(s)) \geq 1, \quad s \in D_R \quad (3.13)$$

on the minimum singular value of  $I+G(s)$ .

To work with  $\hat{G}(s)$  instead of  $G(s)$ , it is necessary to manipulate Fig. 2.6 into the equivalent (for stability analysis) form depicted in Fig. 3.2. Then using (3.6) and (3.7) together with Theorem 2.3 leads directly to the following result.

Theorem 3.2: The polynomial  $\tilde{\phi}_{CL}(s)$  has no CRHP zeroes provided the following conditions are satisfied:

(a) conditions (2.16) and (2.17) hold (3.14)

(b)  $G(s)$  is specified by (3.4) where  $K > 0$  satisfies (3.2) and  $[A, B]$  is stabilizable,  $[A, Q^{\frac{1}{2}}]$  is detectable and  $B$  has full rank. (3.15)

(c) with  $\gamma(s) \triangleq \bar{\sigma}(R^{\frac{1}{2}}L^{-1}(s)R^{-\frac{1}{2}} - I)$  either of the following hold

$$(i) \quad Q > 0 \text{ and } \gamma(s) \leq 1, \quad s \in \Omega_R \quad (3.16)$$

$$(ii) \quad \phi_{0L}(j\omega) \neq 0 \quad \forall \omega \text{ and } \gamma(s) < 1, \quad s \in \Omega_R \quad (3.17)$$

Proof: Appendix

Note that the condition  $\bar{\sigma}(R^{\frac{1}{2}}L(s)R^{-\frac{1}{2}} - I) \leq 1$  in (3.16) can be rewritten as

$$RL(s) + L^H(s)R - R \geq 0, \quad s \in \Omega_R \quad (3.18)$$

or with  $s = j\omega$

$$L(j\omega)R^{-1} + R^{-1}L^H(j\omega) - R^{-1} \geq 0 \quad \forall \omega \quad (3.19)$$

The inequality (3.19) is used by Safonov and Athans [25] to prove the LQ state feedback guaranteed gain and phase margins although their method, of proof is quite different. They implicitly assume that  $L(j\omega)$  is stable, something which we do not require.

Theorem 3.2 can now be employed to establish the guaranteed minimum multivariable gain and phase margins associated with LQ regulators. We emphasize that these margins are guaranteed only if  $R$  is chosen to be a diagonal matrix; we will subsequently present an example showing that the margins can be made arbitrarily small for an appropriately chosen non-diagonal  $R$  matrix.

Corollary 3.1: The LQ regulator with loop transfer matrix  $G(s)$  satisfying (3.13) has simultaneously in each feedback loop a guaranteed minimum gain margin (GM) given by

$$GM = \frac{1}{2}, \quad \infty \quad (3.20)$$

and also a guaranteed minimum phase margin (PM)

$$PM = \pm 60^\circ \quad (3.21)$$

if  $R$  is diagonal and either  $Q > 0$  in (3.2) or  $\phi_{0L}(j\omega) \neq 0, \forall \omega$ .

Proof: Appendix

If  $\alpha_i$  represents a pure gain change or  $\alpha_i = e^{j\phi_i}$  represents a pure phase change, then Fig. 3.3 illustrates the placement of the  $\alpha_i$  in the feedback loop. Note that the interpretation of the gain and phase margins specified in (3.20) and (3.21) changes slightly depending on whether  $Q > 0$  or  $\phi_{0L}(j\omega) \neq 0, \forall \omega$  holds. If  $Q > 0$ , then the pure gain  $\alpha_i$  must satisfy  $\alpha_i \geq 1/2$ . Similarly, for  $\alpha_i = e^{j\phi_i}$  we must have  $|\phi_i| \leq 60^\circ$ . If only  $\phi_{0L}(j\omega) \neq 0, \forall \omega$  holds then pure gains  $\alpha_i$  must satisfy  $\alpha_i > 1/2$  while pure phase factors  $\alpha_i = e^{j\phi_i}$  must satisfy  $|\phi_i| < 60^\circ$ .

Results related to Corollary 3.1 have been derived by various authors [26]-[29]; but the definitive treatment including the multivariable phase margin result is due to Safonov and Athans [25]. The approach of this paper, based on relatively simple frequency domain arguments, is new.

If  $R$  is not diagonal then the guarantees of Corollary 3.1 do not apply. The following example illustrates that the gain margins may become arbitrarily small.

Example 3.1:

Consider the LQ regulator specified by (3.15) when

$$(A, B, Q^{\frac{1}{2}}) = \left( I_2, \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}, I_2 \right) \quad (3.22)$$

and  $R > 0$  is a nondiagonal matrix given by

$$R = B^T [K^2 + 2K] B \quad (3.23)$$

where  $K > 0$  is arbitrary. By selection of  $R$  in (3.23)  $K$  satisfies (3.2). Now let the multiplicative perturbation  $L(s)$  be given by the constant matrix  $L$  where

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 1+\epsilon \end{bmatrix} \quad (3.24)$$

and  $\epsilon \neq 0$  is arbitrary. The zeroes of  $\tilde{\phi}_{CL}(s)$  are the eigenvalues of the perturbed closed-loop system matrix  $\tilde{A}_{CL}$  where

$$\tilde{A}_{CL} = A - BLB^{-1}(2I + K^{-1}) \quad (3.25)$$

or

$$\tilde{A}_{CL} = - \begin{bmatrix} (p_1+2) + \beta \epsilon p_2^{-1} & p_2 + \beta \epsilon (p_3+2) \\ (1+\epsilon)p_2 & (1+\epsilon)(p_3+2) - 1 \end{bmatrix} \quad (3.26)$$

where we have let  $K^{-1}$  be denoted by

$$K^{-1} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \quad (3.27)$$

For  $\tilde{A}_{CL}$  to have no CRHP eigenvalues it is necessary for  $\text{tr } \tilde{A}_{CL} < 0$ . However, by inspection of (3.26), if  $p_2 \neq 0$  then for any  $\epsilon \neq 0$  there exists a  $\beta$  that will make  $\text{tr } \tilde{A}_{CL} > 0$  and therefore for arbitrarily small  $\epsilon$ , the perturbed closed-loop system will be unstable.

The basic problem exposed here is that the margins are really guaranteed at a different point in the loop than where we would like. This is illustrated in Fig. 3.4 where the perturbation  $L(s)$  is inserted at point ①. When  $L(s)$  is diagonal, as when calculating gain and phase margins, and  $R$  is also diagonal then  $R^{-\frac{1}{2}}$  and  $L(s)$  commute and points ① and ② have identical guaranteed gain and phase margins. Point ② is where it is important to have margins (i.e., at the input to the physical plant), not inside the compensator at point ①.

Returning to Example 2.1 of the previous section once more, an LQ feedback control law is given that has the same closed-loop poles as before, but avoids the near instability associated with the negative identity feedback.

Example 3.2:

With  $b_{12} = 50$  in (2.10) as in the plot of  $\underline{G}(I+G(j\omega))$  in Fig. 2.9, an LQ design using  $R=I$  and

$$Q = 3 \begin{bmatrix} 2601 & -50 \\ -50 & 1 \end{bmatrix} \quad (3.28)$$

gives a feedback gain of

$$R^{-1}B^T K = \begin{bmatrix} 1 & -50 \\ 0 & 1 \end{bmatrix} \quad (3.29)$$

and a closed-loop system matrix  $A_{CL}$  of

$$A_{CL} = A - BR^{-1}B^T K = -2I \quad . \quad (3.30)$$

This makes  $I+G(s)$

$$I+G(s) = \begin{bmatrix} \frac{s+2}{s+1} & 0 \\ 0 & \frac{s+2}{s+1} \end{bmatrix} \quad (3.31)$$

and thus

$$\underline{\sigma}(I+G(j\omega)) = \left[ \frac{\omega^2+4}{\omega^2+1} \right]^{\frac{1}{2}} \geq 1 \quad . \quad (3.32)$$

As one might expect the ability of LQ regulators to tolerate crossfeed perturbations is also affected by the choice of the control weighting matrix  $R$ . This is made precise in the following corollary.

Corollary 3.2: The LQ regulator with loop transfer matrix  $G(s)$  satisfying (3.15) will tolerate (i.e.,  $\phi_{CL}(s)$  will have no CRHP zeroes) a crossfeed perturbation of the form (2.42) satisfying (2.16) and (2.17) provided

$$\bar{\sigma}^2(X(s)) < \min \left\{ \frac{\lambda_{\min}(R_1)}{\lambda_{\max}(R_2)}, \frac{\lambda_{\min}(R_2)}{\lambda_{\max}(R_1)} \right\}, \quad s \in \Omega_R \quad (3.33)$$

where  $R$  is given by

$$R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \quad (3.34)$$

and is conformably partitioned with  $L(s)$  in (2.42) and either  $Q > 0$  or  $\phi_{0L}(j\omega) \neq 0 \quad \forall \omega$  holds.

Proof: Appendix

Note that

$$\frac{\lambda_{\min}(R)}{\lambda_{\max}(R)} \leq \min \left\{ \frac{\lambda_{\min}(R_1)}{\lambda_{\max}(R_2)}, \frac{\lambda_{\min}(R_2)}{\lambda_{\max}(R_1)} \right\} \leq \left[ \frac{\lambda_{\min}(R)}{\lambda_{\max}(R)} \right]^{1/2} \quad (3.35)$$

which indicates that if the ratio of  $\lambda_{\min}(R) / \lambda_{\max}(R)$  is very small that the ability to tolerate crossfeed perturbations is drastically reduced. As illustrated in Fig. 3.4 the use of  $R$  scales the inputs and outputs such that the stability margins are obtained in the scaled system rather than the original system. This means that if our original model has the coordinate system in which we would like to guaranteee margins, that  $R$  should be selected as  $R = \rho I$  for some positive scalar  $\rho$ .

Since LQ designs have inherently good margins provided  $R$  is selected appropriately, it is natural to search for variations of this method. One such variation, proposed by Wong and Athans [27], is to solve a Lyapunov rather than a Riccati equation to compute  $K$  in (3.11).

The Lyapunov equation with  $Q \geq 0$  given by

$$A^T K + K A + Q = 0 \quad (3.36)$$

guarantees that the eigenvalues of  $A$  lie in the CLHP if  $K \geq 0$  and  $[A, Q^{1/2}]$  is detectable. The corresponding Kalman type inequality for loop transfer matrices  $G(s)$  specified by (3.4) where  $K \geq 0$  satisfies (3.36) is given by.

$$R G(j\omega) + G^H(j\omega) R \geq 0, \quad \forall \omega \quad (3.37)$$

and is the fundamental inequality used to derive stability margins. When  $Q > 0$  the inequality (3.33) may be changed to strictly greater than. The stability margins for this type of feedback are given in the next theorem

and its corollaries.

Theorem 3.3: For  $G(s)$  of the form of (3.4),  $\tilde{\phi}_{CL}(s)$  has no CRHP zeroes if the following conditions hold:

(a)  $\tilde{\phi}_{0L}(s)$  has no CRHP zeroes (3.38)

(b)  $K \geq 0$  satisfies (3.36) with  $Q \geq 0$ ,  $R > 0$  and  $[A, Q^{1/2}]$  detectable, and  $B$  has full rank (3.39)

(c) either of the following holds

(i)  $Q > 0$  and  $RL(s) + L^H(s)R \geq 0$ ,  $s \in \Omega_R$  (3.40)

(ii)  $RL(s) + L^H(s)R > 0$   $s \in \Omega_R$  (3.41)

Proof: Appendix

Corollary 3.3: For  $G(s)$  as in Theorem 3.3 with  $R$  diagonal the guaranteed gain and phase margins are given by

$$GM = 0, \infty \quad (3.42)$$

and

$$PM = \pm 90^\circ \quad (3.43)$$

Proof: Similar to Corollary 2.3

The importance of Corollary 3.3 is that the standard LQ guaranteed gain reduction margin of  $\frac{1}{2}$  can be reduced to 0 by using  $K$  satisfying the Lyapunov equation (3.36) with  $Q > 0$  rather than the Riccati equation (3.2). Of course, it is possible to have a zero gain reduction margin only for open-loop stable

systems. However, standard LQ state feedback does not guarantee a zero gain reduction margin even in the open-loop stable case, and has been criticized on these grounds [20]. Having a zero gain reduction margin is important in situations where actuators may fail or saturate, and there is no opportunity to reconfigure the control system. In fact, the motivation for the thesis [26] (which in turn lead to most of the developments reported in this paper) was a study supporting the design of the automatic depth-keeping controller for the Trident submarine, in which saturation of one of the two actuators produced an unstable closed-loop system.

Corollary 3.4: For  $G(s)$  as in Theorem 3.3 the crossfeed tolerance is given by

$$\bar{\sigma}^2(X(s)) < 4 \min \left[ \frac{\lambda_{\min}(R_1)}{\lambda_{\max}(R_2)}, \frac{\lambda_{\min}(R_2)}{\lambda_{\max}(R_1)} \right], \quad s \in \Omega_R \quad (3.44)$$

where  $L(s)$  is given by (2.42),  $R > 0$  is given by (3.34) and (3.38) holds ensuring  $\tilde{\Phi}_{CL}(s)$  has no CRHP zeroes.

Proof: Analogous to Corollary 3.2.

Another way to modify the LQ design procedure that is a compromise between Theorem 3.2 and Theorem 3.3 involves using a parameterized Riccati equation given by

$$A^T K + K A + Q - \beta K B R^{-1} B^T K = 0 \quad (3.45)$$

where  $\beta$  is an adjustable parameter and  $0 < \beta < 2$ . The feedback law is still given by (3.11) and  $G(s)$  is still given by (3.4) with  $K \geq 0$ . Since the  $\beta$  in (3.45) may be lumped together with the  $R$  matrix, (3.45) is just a standard

Riccati equation and therefore has a unique solution  $K \geq 0$  under the appropriate assumptions (3.15). The standard LQ optimal feedback law associated with (3.45) is given by

$$u(t) = -\beta R^{-1} B^T K x(t) . \quad (3.46)$$

Instead of (3.46) we will use  $u(t) = -R^{-1} B^T K x(t)$  as in (3.11). Thus depending on whether  $\beta > 1$  or  $\beta < 1$  we are merely decreasing or increasing, respectively, the optimal feedback gain by a scalar factor of  $1/\beta$ . Also with  $G(s)$  given by (3.4) the standard LQ loop transfer matrix is simply  $\beta G(s)$ . From Theorem 3.1 we know that if  $Q > 0$

$$[I + \beta G(s)]^H \frac{1}{\beta} R [I + \beta G(s)] > \frac{1}{\beta} R, \quad s \in \Omega_R \quad (3.47)$$

which in the SISO case becomes

$$\left| \frac{1}{\beta} + g(s) \right| > \frac{1}{\beta}, \quad s \in \Omega_R \quad (3.48)$$

and is illustrated in Fig. 3.5(a). To obtain bounds on  $L(s)$  to ensure stability we merely work with  $\frac{1}{\beta} L(s)$  and  $\beta G(s)$  and apply Theorem 3.2 for the standard LQ regulator problem. Doing this we obtain, in the SISO case, the inequality

$$\left| \beta \ell^{-1}(s) - 1 \right| < 1, \quad s \in \Omega_R, \quad (3.49)$$

illustrated in Fig. 3.5(b). Note that from Fig. 3.5(a) that the critical "-1" point is no longer contained inside the circle if  $\beta > 2$  and thus there are no guaranteed margins. If  $\beta \rightarrow 0$  the guaranteed minimum margins approach those of the Lyapunov feedback case. In general, for the multivariable case the

guaranteed minimum margins, again if  $R$  is diagonal and  $Q$  is positive definite, are given by

$$GM = \beta/2, \infty , \quad 0 < \beta < 2 \quad (3.50)$$

and

$$PM = \pm \cos^{-1} \frac{\beta}{2} , \quad 0 < \beta < 2 . \quad (3.5.1)$$

These guaranteed margins (when  $\beta \leq 1$ ) can also be obtained by similar but distinctly different procedures reported in [30] and [31] which utilize standard LQ regulators with vanishingly small control weights.

### C. Stability Margins of LQG Regulators

A basic limitation associated with the LQ guaranteed stability margins is that they are obtained only under the assumption of full state feedback. State feedback can never be exactly realized, and often it is impossible or too expensive to provide enough sensors to achieve even an approximate realization. Thus one is motivated to investigate what guaranteed stability margins might be associated with LQG controllers, in which a Kalman filter (KF) is used to provide state estimates for feedback.

Since the Kalman filter is the dual of the LQ regulator, dual robustness results are obtainable. They ensure a nondivergent Kalman filter under variations in the nominal model parameters of the plant whose state is to be estimated. To make the precise connection between the regulator and filter problems, consider the linear system

$$\dot{x}(t) = Ax(t) + \xi(t) \quad (3.52)$$

$$y(t) = Cx(t) + \theta(t) \quad (3.53)$$

where  $\xi(t)$  and  $\theta(t)$  are zero mean white noise sources with spectral intensity matrices  $\Xi$  and  $\Theta$  respectively. We wish to estimate  $x(t)$  given  $y(\tau)$ ,  $-\infty \leq \tau \leq t$ , such that the mean square error is minimized. Under the assumption the  $[A, C]$  is detectable, it is well-known that the state estimate is specified by

$$\hat{x}(t) = A\hat{x}(t) + \Sigma C^T \Theta^{-1} v(t) \quad (3.54)$$

$$v(t) = y(t) - C\hat{x}(t) \quad (3.55)$$

where

$$A\Sigma + \Sigma A^T + \Xi - \Sigma C^T \Theta^{-1} C \Sigma = 0, \quad \Sigma \geq 0. \quad (3.56)$$

If we calculate the transfer matrix from  $v(s)$  to  $\hat{y}(s) = C\hat{x}(s)$ , we find that

$$\hat{y}(s) = [C(I_s - A)^{-1} \Sigma C^T \Theta^{-1}] v(s) \triangleq F(s) v(s). \quad (3.57)$$

Then, if  $\Xi > 0$ ,  $F(s)$  satisfies the dual of (3.6) given by

$$(I + F(s)) \Theta (I + F(s))^H > 0, \quad s \in \Omega_R \quad (3.58)$$

which guarantees the stability of the error dynamics under a range of perturbations in  $F(s)$ . Thus, if  $F(s)$  is perturbed to  $\tilde{F}(s) = F(s)L(s)$ , where usual assumptions about  $G(s)$  are applied to  $F(s)$ , the Kalman filter will remain nondivergent if

$$\tilde{\sigma}(\Theta^{-1/2} L^{-1}(s) \Theta^{1/2} - I) \leq 1, \quad s \in \Omega_R \quad (3.59)$$

or equivalently,

$$\theta L^H(s) + L(s)\theta - \theta \geq 0. \quad (3.60)$$

It is now readily apparent that  $F(s)$ , the loop transfer matrix of the error dynamics loop of the Kalman filter, is the dual of  $G(s)$  in the LQ regulator and has the same guaranteed margins at its input,  $v(s)$ , for diagonal  $\theta$ .

Safonov and Athans [32] have developed these dual results for the nondivergence of the extended Kalman filter. Furthermore, they have considered the robustness properties of a nonlinear LQG control system formed by the cascade of a constant gain extended Kalman filter and the LQ state feedback gain. The LQ state feedback gain and the constant gain of the extended Kalman filter are computed from the linearized model parameters. However, the extended Kalman filter must have the true nonlinear model of the plant. In the completely linear case the LQG stability margins are much easier to obtain.

The standard LQG control system block diagram is shown in Fig. 3.6. with various points of the loop marked. To determine the robustness of the LQG control system we insert perturbations at points ② and ③ (the input and output of the physical plant) and find out how large they can be made without destabilizing the closed-loop system. It is therefore convenient to calculate the loop transfer matrices at points ① to ④. The loop transfer matrix at point ⑤ will be denoted  $T_K(s)$  and is calculated breaking the loop at point ⑤ and using it as the input as well as the output. For the four points indicated in Fig. 3.6 we have

$$T_1(s) = G_r \phi(s) B \quad (3.61)$$

$$T_2(s) = G_r (\phi^{-1}(s) + BG_r + G_f C)^{-1} G_f C \phi(s) B \quad (3.62)$$

$$T_3(s) = C\phi(s)B G_r (\phi^{-1}(s) + BG_r + G_f C)^{-1} G_f \quad (3.63)$$

$$T_4(s) = C\phi(s)G_f \quad (3.64)$$

where

$$G_r \triangleq R^{-1} B^T K = \text{regulator gain} \quad (3.65)$$

$$G_f \triangleq \Sigma C^T \theta^{-1} = \text{filter gain} \quad (3.66)$$

$$\phi(s) \triangleq (Is - A)^{-1} \quad (3.67)$$

Note that points ① and ④ have the standard LQ regulator and Kalman filter loop transfer matrices respectively. Thus at points ① and ④ (inside the LQG controller) the LQ and KF minimum guaranteed stability margins apply. The following theorem is a much simplified version of a theorem proved in [32] and gives LQG stability margins at points ② and ③ (the input and output of the physical plant).

Theorem 3.4: The LQG feedback control system of Fig. 3.6 is asymptotically stable under variations in the open-loop plant  $G_p(s) \triangleq C(Is - A)^{-1}B$  if the following conditions hold:

(a) the perturbed open-loop plant  $\tilde{G}_p(s) \triangleq \tilde{C}(Is - \tilde{A})^{-1}\tilde{B}$  is such that  $\det(sI - \tilde{A})$  and  $\det(sI - A)$  have the same number of CRHP zeroes and if  $\det(j\omega_0 I - \tilde{A}) = 0$  then  $\det(j\omega_0 I - A) = 0$ . (3.69)

(b)  $[A, B]$  is stabilizable,  $Q > 0$ ,  $R > 0$  and  $K \geq 0$  satisfies (3.2) and  $B$  has full rank. (3.70)

$$(d) \tilde{G}_p(s) = G_p(s)L(s) = N(s)G_p(s) \quad (3.71)$$

and either

$$\bar{\sigma}(R^{1/2}L^{-1}(s)R^{-1/2} - I) \leq 1 \quad (3.72)$$

or

$$\bar{\sigma} (\theta^{-1/2} N^{-1}(s) \theta^{1/2} - 1) \leq 1 \quad (3.73)$$

hold for all  $s \in \Omega_R$ .

(e) the LQG controller transfer matrix  $G_C(s)$  from the plant output to the plant input is given by

$$G_C(s) = G_r (I - \tilde{A} + \tilde{B}G_r + G_f \tilde{C})^{-1} G_f \quad (3.74)$$

where  $G_r$  and  $G_f$  respectively satisfy (3.65) and (3.66).

Proof: Appendix

Notice that in (3.71)  $L(s)$  represents the same perturbation in  $G_p(s)$  at the input to the plant as  $N(s)$  represents at the output of the plant and that  $\tilde{G}_p(s)$  is the same in both cases. Now the basic idea of the proof is quite simple. At point (1) we have an LQ state feedback regulator loop transfer matrix and the LQ guaranteed margins apply. By moving  $L(s)$ , the perturbation, to point (2) we simply change  $B$  in the Kalman filter to  $BL(s)$  leaving  $G_r = R^{-1}B^T K$  fixed. This, however, is the same as giving the Kalman filter the correct dynamic model of the perturbed open-loop system without changing either the filter or the regulator gains. The same result follows if we start with a perturbation,  $N(s)$ , at point (4), where the KF guarantees apply and move it to point (3) changing  $C$  to  $CN(s)$ .

Thus the LQ and KF guaranteed stability margins will apply to LQG controllers at the input and output of the physical plant but under the restrictive assumption that the system model embedded within the Kalman filter is always the same as the true system. For the more realistic case in which the internal model of the Kalman filter remains unchanged, there are

unfortunately no guarantees, as Doyle has demonstrated with a simple counterexample [33]. This counterexample is extreme, but it is possible to obtain LQG controllers with inadequate stability margins that look quite reasonable in the time domain. Fig. 3.7 shows the Nyquist plot of a single-input design reported in the literature [34]; note that the phase margin is less than 10°.

Fortunately, there are two dual procedures that do not require the Kalman filter to have the true system model and that still recover the LQ and KF guaranteed minimum margins. These procedures use the asymptotic properties of the Kalman filter and LQ regulator (see [ ] this issue) and can be used only if the plant is minimum phase. If  $W$  is a nonsingular arbitrary matrix, then by selecting  $E$  in (3.56) as  $\rho BWW^T B^T$  and letting  $\rho \rightarrow \infty$  the loop transfer matrix  $T_2(s)$  in (3.62) approaches  $T_1(s)$  of (3.61) if the minimum phase assumption holds [35]. Thus the LQ regulator guaranteed margins will be recovered at the input to the plant. Kwakernaak [36] proposed the dual of the above procedure to obtain low sensitivity feedback systems. His procedure makes  $T_3(s)$  of (3.63) approach  $T_4(s)$  of (3.64) and thus the KF guaranteed minimum margins will be recovered at the output of the plant<sup>1</sup>. However, it is not always the case that an LQG controller needs to be robustified by these procedures since in some cases the LQG control system will have better stability margins than its full state feedback counterpart [42].

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<sup>1</sup>Dowdle [37] has adapted these procedures for use with minimal order observer based compensators and their duals.

Even when these procedures are used, the guaranteed stability margins apply at the input or output of the physical plant but not necessarily at both input and output. It is desirable to have margins at both these locations since the perturbations in  $G_p(s)$  are represented as either  $G_p(s)L(s)$  or  $N(s)G_p(s)$  and we should not like small perturbations in either input or output to destabilize the system. Margins at both input and output can be ensured if the inequalities

$$\underline{\sigma}(I+G_c(s)G_p(s)) \geq 1 \quad (3.75)$$

and

$$\underline{\sigma}(I+G_p(s)G_c(s)) \geq 1 \quad (3.76)$$

both hold. The relationship between these two quantities when  $G_p(s)$  and  $G_c(s)$  are square is given by

$$\frac{1}{k} \underline{\sigma}(I+G_p(s)G_c(s)) \leq \underline{\sigma}(I+G_c(s)G_p(s)) \leq k \underline{\sigma}(I+G_p(s)G_c(s)) \quad (3.77)$$

where

$$k = \min \left[ \frac{\overline{\sigma}(G_p(s))}{\underline{\sigma}(G_p(s))}, \frac{\overline{\sigma}(G_c(s))}{\underline{\sigma}(G_c(s))} \right] \geq 1. \quad (3.78)$$

The quantity  $k$  is the minimum of the condition numbers of  $G_p(s)$  and  $G_c(s)$  with respect to inversion. The proof is accomplished by a simple calculation and is omitted. From (3.77) we conclude that if  $k$  is close to unity then approximately the same robustness guarantees will apply at both input and output. Note that we have no control over  $G_p(s)$  so that

if  $G_p(s)$  is nearly singular we must design our compensator so that  $\bar{\sigma}(G_c(s)) \approx \underline{\sigma}(G_c(s))$ . On the other hand, if our plant is well-conditioned with respect to inversion our compensator  $G_c(s)$  need not be so severely constrained, allowing more flexibility in achieving performance objectives.

#### IV. SUMMARY AND CONCLUSIONS

##### A. Summary

In this paper we have stressed the importance of the robustness of feedback systems with respect to uncertainty in the nominal plant model. Arguing from the multivariable Nyquist theorem, it was shown that if the return difference matrix  $I+G(s)$  is nearly singular, then there exists a small perturbation in  $G(s)$  that will destabilize the closed-loop system. To detect this condition, singular values, familiar from numerical analysis, were introduced. In particular  $\underline{\sigma}(I+G(s))$  measures the nearness to singularity of  $I+G(s)$ . Interpreting  $\underline{\sigma}(I+G(s))$  lead to a direct multivariable generalization of the classical notions gain and phase margins. These margins were shown to hold simultaneously in all loops of the feedback system. Also a channel crossfeed margin was derived from  $\underline{\sigma}(I+G(s))$  which is the key quantity in the determination of all the stability margin results.

Direct synthesis procedures involving  $\underline{\sigma}(I+G(s))$  are unknown at present. This is due in part to the complicated dependence of  $\underline{\sigma}(I+G(s))$  on the compensator implicit in  $G(s)$ . However, the LQ state feedback regulator was shown to provide an indirect synthesis procedure which automatically ensures a degree of robustness in the coordinate system specified by the control weighting matrix  $R$ . Thus LQ state feedback is preferable to state feedback specified by pole placement techniques since there are no robustness guarantees provided by this latter method. Next, using the Lyapunov equation to compute the feedback gains, a regulator was specified that was stable in spite of the failure of any of its feedback channels. For LQG control systems there are no

automatic guarantees. However, in the case of minimum phase plants, the LQ-guaranteed margins are asymptotically recoverable if necessary.

Even so, these guarantees will simply be inadequate if excessively large feedback gains are used. This is most clear in the SISO case when the gain crossover frequency occurs at a frequency at which the phase of the open-loop transfer function is completely uncertain. In this case the phase margin necessary for stability is  $\pm 180^\circ$  requiring the loop gain to be less than 1.

#### B. Conclusions and Future Research

Although the singular value approach is useful in detecting the near instability of a control system, it is sometimes unnecessarily conservative. This is due to the fact that some of the small perturbations that would theoretically destabilize the closed-loop system will never occur in the physical system. Nevertheless they are still detected by a small  $\underline{\sigma}(I+G(s))$ . One direction for further research is characterization of the robustness of a feedback system in which perturbations in certain directions are ruled out as impossibilities on physical grounds. This points out the fundamental problem of obtaining the characterization of the uncertainty associated with a given model. It seems that this knowledge can be acquired only by experience with real applications. One of the advantages of the singular value approach is that it singles out the worst type of perturbations for scrutiny. Another avenue for research is, of course, LQG controller design procedures that ensure a minimum size for  $\underline{\sigma}(I+G(s))$  or its complementary quantity  $\underline{\sigma}(I+G^{-1}(s))$  and yet maintain a satisfactory degree of performance.

In the decade since the promulgation of the LQG methodology [41], much has been learned about the pitfalls of its application to practical problems. It has been criticized on the grounds that the optimal control problem is over specified to the extent that there is only one solution merely to be found by the computer [1]. Practice has shown however, that the LQG method is not merely a "cookbook" procedure to be used blindly, but one in which a fair amount of iteration on the design of a controller is necessary to obtain satisfactory results. Unlike, however, many of the frequency domain techniques which reduce a multiloop problem to a series of single loop designs, the LQG procedure is inherently a multiloop procedure. It provides a reasonable place to start a control system design and when used intelligently should provide the designer a good chance of success. .

## APPENDIX

### A. Proofs for Sections II and III

Proof of Theorem 2.2: For all  $R$  sufficiently large,  $D_R$  will enclose all ORHP zeroes of both  $\phi_{OL}(s)$  and  $\tilde{\phi}_{OL}(s)$  and by virtue of (2.17) and the indentation construction of  $D_R$  this can be extended to all CRHP zeroes of both  $\phi_{OL}(s)$  and  $\tilde{\phi}_{OL}(s)$ . Also, for  $R$  sufficiently large,  $D_R$  avoids all OLHP zeroes of  $\phi_{OL}(s)$ ,  $\tilde{\phi}_{OL}(s)$  and  $\phi_{CL}(s)$ . From Theorem 2.1 and (2.18) we conclude that

$$\mathcal{N}(0, \det[I+G(s)], D_R) = -P \quad (A.1)$$

where  $P$  is the number of CRHP zeroes of  $\phi_{OL}(s)$  and, by (2.16), also of  $\tilde{\phi}_{OL}(s)$ . Clearly  $\det[I+(1-\epsilon)G(s) + \epsilon \tilde{G}(s)]$  is a continuous function of  $\epsilon$  for all  $s \in D_R$  and from (A.1)

$$\mathcal{N}(0, \det[I+(1-\epsilon)G(s) + \epsilon \tilde{G}(s)], D_R) \Big|_{\epsilon=0} = -P \quad (A.2)$$

Now suppose that as  $\epsilon$  is varied continuously from 0 to 1 that

$\mathcal{N}(0, \det[I+(1-\epsilon)G(s) + \epsilon G(s)], D_R)$  does not remain constant at  $-P$ . From the Principle of the Argument we know that for some  $\epsilon$  on  $[0, 1]$  the number of zeroes minus the number of poles of  $\det[I+(1-\epsilon)G(s) + \epsilon G(s)]$  enclosed in  $D_R$  must change. However, since the poles and zeroes are also continuous functions of  $\epsilon$ , it must be that for some  $\epsilon_0$  on  $[0, 1]$  that they lie on  $D_R$  and thus

$$\det[I+(1-\epsilon_0)G(s) + \epsilon_0 \tilde{G}(s)] = 0 \text{ or } \infty \quad (A.3)$$

Condition (2.19) eliminates the possibility that  $\det[I+(1-\epsilon_0)G(s) + \epsilon_0 \tilde{G}(s)]$  evaluates to zero. Since  $R$  is chosen sufficiently large,  $D_R$  must avoid all

zeros of  $\tilde{\phi}_{0L}(s)$  and  $\phi_{0L}(s)$  which include the poles of  $\det[I + (I - \epsilon_0)G(s) + \epsilon_0 \tilde{G}(s)]$  and thus the possibility that  $\det[I + (1 - \epsilon_0)G(s) + \epsilon_0 \tilde{G}(s)]$  is infinite is also ruled out. This contradicts the assumption that  $\mathcal{M}(0, \det[I + (I - \epsilon)G(s) + \epsilon G(s)], D_R)$  changes as  $\epsilon$  is varied on  $[0, 1]$  and therefore it must be that it remains constant for all  $\epsilon$  on  $[0, 1]$ . However, this implies that for  $\epsilon = 1$  that

$$\mathcal{M}(0, \det[I + \tilde{G}(s)], D_R) = -P \quad (A.4)$$

and thus by (2.16) and Theorem 2.1,  $\tilde{\phi}_{CL}(s)$  has no CRHP zeroes.

Q.E.D.

Lemma A.1: For square matrices  $G$  and  $L$   $\det(I + GL) \neq 0$  if either of the following conditions hold:

$$(a) \quad \bar{\sigma}(L^{-1} - I) < \underline{\sigma}(I + G) \quad (A.5)$$

$$(b) \quad G + G^H \geq 0 \quad \underline{\text{and}} \quad (A.6)$$

$$L + L^H > 0 \quad (A.7)$$

Proof: To prove (a) rewrite  $I + GL$  as

$$I + GL = [(L^{-1} - I)(I + G)^{-1} + I](I + G)L \quad (A.8)$$

since  $L$  and  $I + G$  are, by (A.5), assumed to be nonsingular,  $I + GL$  is nonsingular if and only if  $[(L^{-1} - I)(I + G)^{-1} + I]$  is nonsingular. Condition (A.5) guarantees that

$$\|(L^{-1} - I)(I+G)^{-1}\|_2 < 1 \quad (A.9)$$

which ensures that  $[(L^{-1} - I)(I+G)^{-1} + I]$  is invertible. To prove (b), suppose, contrary to what we wish to prove that

$$\det(I+GL) = 0 \quad (A.10)$$

Thus  $\exists$  a vector  $x \neq 0 \ni (I+GL)x = 0$  and hence

$$x = -GLx \quad (A.11)$$

Defining  $z = Lx$ , (A.11) implies  $z \neq 0$  and

$$z = -LGz \quad (A.12)$$

Condition (A.6) and (A.12) imply

$$z^H G z + z^H G^H z = -z^H G^H [L + L^H] G z \geq 0 \quad (A.13)$$

and since  $Gz \neq 0$  a contradiction to (A.7) and (A.10) is obtained.

Q.E.D.

Lemma A.2: If  $L$  is a square matrix and  $L(\epsilon) = (1-\epsilon)I + \epsilon L$ , then for all  $\epsilon$  on  $[0, 1]$  the following implications hold:

$$(a) \quad \bar{\sigma}(L^{-1} - I) < \alpha \leq 1 \Rightarrow \bar{\sigma}(L^{-1}(\epsilon) - I) < \alpha \quad (A.14)$$

$$(b) \quad \sigma(L^{-1} - I) < \alpha \quad \text{and} \quad L + L^H \geq 0 \Rightarrow \bar{\sigma}(L^{-1}(\epsilon) - I) < \alpha \quad (A.15)$$

$$\begin{aligned}
 (c) \quad \bar{\sigma}(L^{-1} - I) &< \alpha \quad \text{and} \quad 2(\alpha^2 - 1)\underline{\sigma}^2(L - I) > \alpha^2 \bar{\sigma}^2(L + L^H - 2I) \\
 &\Rightarrow \bar{\sigma}(L^{-1}(\epsilon) - I) < \alpha
 \end{aligned} \tag{A.16}$$

$$(d) \quad L + L^H > 0 \Rightarrow L(\epsilon) + L^H(\epsilon) > 0 \tag{A.17}$$

Proof: For (a) and (b) rewrite  $\bar{\sigma}(L^{-1}(\epsilon) - I) < \alpha$  as

$$\alpha^2 L^H(\epsilon) L(\epsilon) - (L(\epsilon) - I)^H (L(\epsilon) - I) > 0 \quad . \tag{A.18}$$

Expanding the left-hand side of (A.18) gives

$$\begin{aligned}
 \alpha^2 L^H(\epsilon) L(\epsilon) - (L(\epsilon) - I)^H (L(\epsilon) - I) &= \epsilon^2 [\alpha^2 L^H L - (L - I)^H (L - I)] + \\
 &+ \alpha^2 (1 - \epsilon) [(1 - \epsilon) I + \epsilon (L + L^H)] \quad .
 \end{aligned} \tag{A.19}$$

By (A.14) or (A.15) the first term on the right-hand side of (A.19) is assumed to be positive definite. Now with  $\alpha \leq 1$ , (A.14) also gives

$$L^H + L > I + (1 - \alpha^2) L^H L > 0 \tag{A.20}$$

so that (A.14) or (A.15) guarantee  $L + L^H \geq 0$  which makes the second term on the right-hand side of (A.19) positive semi-definite. Thus (A.18) is true for all  $\epsilon$  on  $[0, 1]$  which is equivalent to  $\bar{\sigma}(L^{-1}(\epsilon) - I) < \alpha$ . To prove (c) rewrite (A.19) as

(A.21)

$$\alpha^2 L^H(\epsilon) L(\epsilon) - (L(\epsilon) - I)^H (L(\epsilon) - I) = (\alpha^2 - 1) \epsilon^2 (L - I)^H (L - I) + \alpha^2 \epsilon [L + L^H - 2I] + \alpha^2 I \quad .$$

For the right-hand-side of (A.21) to be positive definite, it is sufficient to show

that  $f(\epsilon) > 0$  where  $f(\epsilon)$  is given by

$$f(\epsilon) = \beta^2 \epsilon^2 - 2\phi\epsilon + \alpha^2 \quad (A.22)$$

and where

$$\beta^2 = (\alpha^2 - 1) \underline{\sigma}^2 (L - I) \quad (A.23)$$

$$\phi = \alpha^2 \bar{\sigma} \left( \frac{L+L^H}{2} - I \right) \quad (A.24)$$

Differentiating  $f(\epsilon)$  to find an  $\epsilon^*$  such that  $f(\epsilon^*)$  is minimum results in

$$\epsilon^* = \frac{\phi}{\beta^2} \quad (A.25)$$

and

$$f(\epsilon^*) = \alpha^2 - \frac{\phi^2}{\beta^2} \quad (A.26)$$

Thus  $f(\epsilon^*)$  will be positive if

$$\alpha^2 \beta^2 > \phi^2 \quad (A.27)$$

or

$$2(\alpha^2 - 1) \underline{\sigma}^2 (L - I) > \alpha^2 \bar{\sigma}^2 (L + L^H - 2I) \quad (A.28)$$

which proves (C).

Implication (d) follows trivially by direct substitution.

Q.E.D.

Proof of Theorems 2.3 and 2.4: Conditions (2.21) and (2.44) ensure that only condition (2.19) of Theorem 2.2 need be verified. Since  $\bar{\sigma}[G(s)]$  and

$\bar{\sigma}[\tilde{G}(s)] \rightarrow 0$  as  $|s| \rightarrow \infty$ ,  $\det[1 + (1-\epsilon) G(s) + \epsilon \tilde{G}(s)] \neq 0$  for all  $|s| = R$  for  $R$  sufficiently large. Thus (2.19) need only be checked for  $s \in \Omega_R$ . Lemmas A.1 and A.2 applied at every  $s \in \Omega_R$ , guarantee that (2.19) holds for  $s \in \Omega_R$  and thus for  $s \in D_R$ . The conditions of Theorem 2.2 are satisfied and thus  $\tilde{\phi}_{CL}(s)$  has no CRHP zeroes.

Q.E.D.

Proof of Corollaries 2.1 and 2.3: In Theorems 2.3 and 2.4 take  $L(s)$  to be a diagonal matrix given by

$$L(s) = \text{diag}[\ell_1(s), \ell_2(s), \dots, \ell_n(s)] \quad (A.29)$$

which simplifies condition (2.22) and (2.23) to

$$|\ell_i^{-1}(s) - 1| < \alpha_0 \quad \forall i \quad (A.30)$$

and condition (2.46) to

$$\text{Re}[\ell_i(s)] > 0 \quad \forall i \quad (A.31)$$

To obtain gain margins, let

$$\ell_i(s) = \ell_i, \quad \ell_i \text{ real}$$

then (A.30) becomes

$$\frac{1}{1+\alpha_0} < \ell_i < \frac{1}{1-\alpha_0} \quad (A.32)$$

and (A.31) becomes

$$\ell_i > 0 \quad (A.33)$$

Similarly, to obtain phase margins, let

$$\ell_i(s) = e^{j\phi_i(s)}, \quad \phi_i(s) \text{ real}$$

then (A.30) becomes

$$1 - \frac{\alpha_0^2}{2} < \cos \phi_i(s) \quad (A.34)$$

or

$$|\phi_i(s)| < \cos^{-1} \left[ 1 - \frac{\alpha_0^2}{2} \right] \quad (A.35)$$

while (A.31) becomes

$$\cos \phi_i(s) > 0 \quad (A.36)$$

or

$$|\phi_i(s)| < \pm 90^\circ \quad (A.37)$$

Q.E.D.

Proof of Corollaries 2.2 and 2.4: Only conditions (2.22) and (2.46) need verification. With  $L(s)$  given by (2.42) we have

$$L^{-1}(s) - I = \begin{bmatrix} 0 & -X(s) \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 \\ -X(s) & 0 \end{bmatrix} \quad (A.38)$$

and thus

$$\bar{\sigma}(L^{-1}(s) - I) = \bar{\sigma}(X(s)) < \alpha_0 \leq \underline{\sigma}(I + G(s)) \quad . \quad (A.39)$$

Also,

$$L(s) + L^H(s) = \begin{bmatrix} 2I & X(s) \\ X^H(s) & 2I \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2I & X^H(s) \\ X(s) & 2I \end{bmatrix} \quad (A.40)$$

so that

$$\begin{aligned} \lambda_{\min}(L(s) + L^H(s)) &\geq 2 - \lambda_{\max} \left( \begin{bmatrix} 0 & X(s) \\ X^H(s) & 0 \end{bmatrix} \right) \quad (A.41) \\ &\geq 2 - \bar{\sigma} \left( \begin{bmatrix} 0 & X(s) \\ X^H(s) & 0 \end{bmatrix} \right) = 2 - \bar{\sigma}(X(s)) > 0 \quad . \end{aligned}$$

Q.E.D.

Proof of Theorem 3.1: Direct manipulation of (3.2) gives

$$(s^*I - A^T)K + K(sI - A) - KBR^{-1}B^T K = (Q + 2\text{Re}(s)K) \quad (A.42)$$

where  $s^*$  denotes the complex conjugate of  $s$ . Premultiplying and postmultiplying (A.42) by  $[(sI - A)^{-1}B]^H$  and  $[(sI - A)^{-1}B]$  respectively we obtain

$$RG(s) + G^H(s)R + G^H(s)RG(s) = H(s) \quad (A.43)$$

Adding  $R$  to both sides of (A.43) gives (3.3). Now  $Q + 2\text{Re}(s)K$  will be positive semidefinite for  $s \in D_R$  if  $Q > 0$  and the indentations of  $\Omega_R$  are sufficiently small or if  $\text{Re}(s) \geq 0$ ,  $s \in D_R$  which happens if  $\det(j\omega I - A) \neq 0 \quad \forall \omega$ . Thus

under these conditions  $H(s) > 0$  or  $H(s) \geq 0$  respectively for all  $s \in \Omega_R$ .

Q.E.D.

Proof of Theorem 3.2: It is well-known that condition (3.15) ensures that  $\phi_{CL}(s)$  has no CRHP zeroes. Defining  $\hat{G}(s) \triangleq R^{1/2}G(s)R^{-1/2}$ , we see that  $G(s)$  has a state-space realization  $(A, BR^{-1/2}, R^{-1/2}B^T K)$  and thus its open- and closed-loop characteristic polynomials  $\hat{\phi}_{OL}(s)$  and  $\hat{\phi}_{CL}(s)$  are identical to those of  $(A, B, R^{-1}B^T K)$ . Thus any assumptions about  $\phi_{OL}(s)$  and  $\phi_{CL}(s)$  obviously apply to  $\hat{\phi}_{OL}(s)$  and  $\hat{\phi}_{CL}(s)$ . Similarly, by defining  $\hat{L}(s) \triangleq R^{1/2}L(s)R^{-1/2}$ , we may work with  $\hat{G}(s)$  and  $\hat{L}(s)$  instead of  $G(s)$  and  $L(s)$ . The conditions (3.6) and (3.7) of Theorem 3.1 are equivalent to  $\underline{\sigma}(I + \hat{G}(s)) > 1$  and  $\underline{\sigma}(I + \hat{G}(s)) \geq 1$  respectively. The condition (3.16) and Theorem 3.1 require that

$$\bar{\sigma}(\hat{L}^{-1}(s) - I) \leq 1 < \underline{\sigma}(I + \hat{G}(s)), \quad s \in \Omega_R \quad (A.44)$$

and by Theorem 2.3 we conclude that  $\tilde{\phi}_{CL}(s)$  has no CRHP zeroes. Alternatively condition (3.17) and Theorem 3.1 require that

$$\bar{\sigma}(\hat{L}^{-1}(s) - I) < 1 \leq \underline{\sigma}(I + \hat{G}(s)) \quad (A.45)$$

which again by Theorem 2.3 means  $\tilde{\phi}_{CL}(s)$  has no CRHP zeroes.

Q.E.D.

Proof of Corollary 3.1: From Theorem 3.1 and 3.2 we know if  $Q > 0$  then

$$\bar{\sigma}(R^{1/2}L^{-1}(s)R^{-1/2} - I) = \bar{\sigma}(L^{-1}(s) - I) \leq 1, \quad s \in \Omega_R \quad (A.46)$$

to satisfy (3.16) when  $L(s)$  and  $R$  are diagonal. If  $\phi_{CL}(j\omega) \neq 0 \quad \forall \omega$  then

$$\bar{\sigma}(L^{-1}(s) - I) < 1, \quad s \in \Omega_R \quad (A.47)$$

to satisfy (3.17) when  $L(s)$  and  $R$  are diagonal. The remainder of the proof is completely analogous to Corollary 2.1

Q.E.D.

Proof of Corollary 3.2: Only conditions (3.16) and (3.17) of Theorem 3.2 need to be verified for the  $L(s)$  of (2.42) the rest are satisfied by assumption. Note that for  $s \in \Omega_R$

$$\begin{aligned} \bar{\sigma}(R^{1/2}L^{-1}(s)R^{-1/2}-I) &= \bar{\sigma}\left(\begin{bmatrix} 0 & -R_1^{1/2}X(s)R_2^{-1/2} \\ 0 & 0 \end{bmatrix}\right) \text{ or} \\ &\bar{\sigma}\left(\begin{bmatrix} 0 & 0 \\ -R_2^{1/2}X(s)R_1^{-1/2} & 0 \end{bmatrix}\right) \quad (A.48) \end{aligned}$$

$$\leq \bar{\sigma}(X(s)) \max\{\bar{\sigma}(R_1^{1/2})\bar{\sigma}(R_2^{-1/2}), \bar{\sigma}(R_2^{1/2})\bar{\sigma}(R_1^{-1/2})\}$$

and hence if

$$\bar{\sigma}(X(s)) \max\left[\frac{\lambda_{\max}(R_1^{1/2})}{\lambda_{\min}(R_2^{1/2})}, \frac{\lambda_{\max}(R_2^{1/2})}{\lambda_{\min}(R_1^{1/2})}\right] < 1 \quad (A.49)$$

then conditions (3.16) and (3.17) are both satisfied. However, (A.49) is equivalent to (3.33).

Q.E.D.

Proof of Theorem 3.3: Conditions (3.38) and (3.39) and the Lyapunov stability criterion guarantee that condition (2.44) of Theorem 2.4 is satisfied.

As in the proof of Theorem 3.2 we may work with  $\hat{G}(s) = R^{1/2}G(s)R^{-1/2}$  and  $\hat{L}(s) = R^{1/2}L(s)R^{-1/2}$  instead of  $G(s)$  and  $L(s)$ . Condition (3.37) is simply condition (2.45) of Theorem 2.4 with  $\hat{G}(s)$  replacing  $G(s)$  and  $\hat{L}(s)$  replacing  $L(s)$  in (2.46) is simply (3.41). Thus by Theorem 2.4 the theorem is proved when (3.41) holds. When  $Q > 0$  and (3.40) is satisfied, the strictness of the inequality (2.46) of Theorem 2.4 may be changed to  $\geq$  and the  $\leq$  of (2.45) to  $>$  and Theorem 2.4 remains valid. Thus when (3.40) holds Theorem 3.1 is again proved.

Q.E.D.

Proof of Theorem 3.4: Breaking the loop at point (1) of Fig. 3.6 we have a loop transfer function matrix of

$$G_r(sI - A + G_f C)^{-1} [G_f C (sI - A)^{-1} B + B] = G_r(sI - A)^{-1} B \triangleq G(s) \quad (A.50)$$

so that

$$\phi_{0L}(s) = \det[sI - A + G_f C] \det[sI - A] \quad (A.51)$$

and

$$\tilde{\phi}_{0L}(s) = \det[sI - A + G_f C] \det[sI - \tilde{A}] \quad (A.52)$$

Since the Kalman filter error dynamics are stable given (3.70) and since (3.68) holds, conditions (2.16) and (2.17) of Theorem 2.2 hold. Now by direct application of Theorem 3.2 we conclude that the system of Fig. 3.5 is stable if  $L(s)$  is inserted at point (1). However, this is not the location we desire to have the margins guaranteed. Nevertheless, by manipulation of the block diagram of Fig. 3.6 we may place  $L(s)$  at point (2) if we change  $B$  to  $BL(s)$  inside the controller leaving  $G_r = R^{-1}B^T K$  fixed. This, however, is equivalent

to changing  $(A, B, C)$  to  $(\tilde{A}, B, C)$  inside the controller leaving  $G_f$  and  $G_r$  fixed which is the desired result for perturbation  $L(s)$ . The proof for a perturbation  $N(s)$  at the output is analogous.

Q.E.D.

**B. Singular Values [19], [38] - [40]**

The singular values of a square  $n \times n$  complex matrix  $A$ , denoted  $\sigma_i(A)$ , are defined as

$$\sigma_i(A) \triangleq \lambda_i^{1/2}(A^H A) = \lambda_i^{1/2}(A A^H) \quad (B.1)$$

where  $A^H$  denotes the complex conjugate transpose of  $A$  and  $\lambda_i(A^H A)$  the  $i^{\text{th}}$  largest eigenvalue of  $A^H A$ . A way of representing the matrix  $A$ , known as the singular value decomposition (SVD) is given by

$$A = U \Sigma V^H = \sum_{i=1}^n \sigma_i(A) \underline{u}_i \underline{v}_i^H \quad (B.2)$$

where

$$U \triangleq [\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n] ; \quad U^H U = I \quad (B.3)$$

$$V \triangleq [\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n] ; \quad V^H V = I \quad (B.4)$$

$$\Sigma = \text{diag}[\sigma_1, \sigma_2, \dots, \sigma_n] \quad (B.5)$$

and the columns of  $V$  and  $U$  are eigenvectors of  $A^H A$  and  $A A^H$  respectively. The minimum and maximum singular values denoted  $\underline{\sigma}$  and  $\overline{\sigma}$  respectively are sometimes equivalently defined in terms of the spectral matrix norm  $\|\cdot\|_2$  as

$$\bar{\sigma}(A) = \max_{\|x\|_2 \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \|A\|_2 \quad (B.6)$$

and

$$\underline{\sigma}(A) = \min_{\|x\|_2 \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \begin{cases} \|A^{-1}\|_2^{-1}, & \text{if } \det A \neq 0 \\ 0, & \text{if } \det A = 0 \end{cases} \quad (B.7)$$

The minimum singular value  $\underline{\sigma}(A)$  provides a measure of the nearness to singularity of the matrix  $A$  in the following sense. If  $A+E$  is singular then

$$\|E\|_2 = \bar{\sigma}(E) \geq \underline{\sigma}(A) \quad (B.8)$$

Other facts involving singular values that are useful for manipulation follow. The inequality

$$\underline{\sigma}(A) > \bar{\sigma}(B) \quad (B.9)$$

may also be written

$$A^H A > B^H B \quad (B.10)$$

Also, since  $\bar{\sigma}(\cdot)$  is the same as  $\|\cdot\|_2$ , the triangle inequality

$$\bar{\sigma}(A+B) \leq \bar{\sigma}(A) + \bar{\sigma}(B) \quad (B.11)$$

holds. Finally, if  $A^{-1}$  exists then (B.6) and (B.7) give

$$\underline{\sigma}(A) = \frac{1}{\bar{\sigma}(A^{-1})} \quad (B.12)$$

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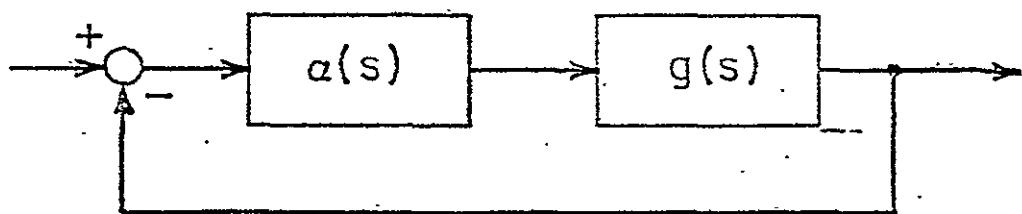


Fig. 1.1: Feedback system for stability margin definition.

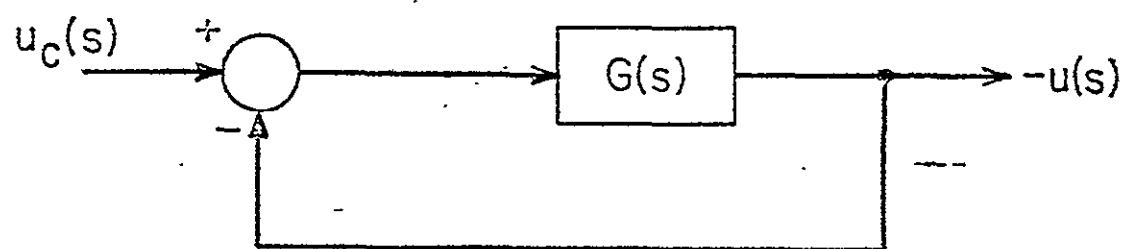


Fig. 2.1: Feedback system where  $G(s)$  represents the open-loop plant plus a compensator.

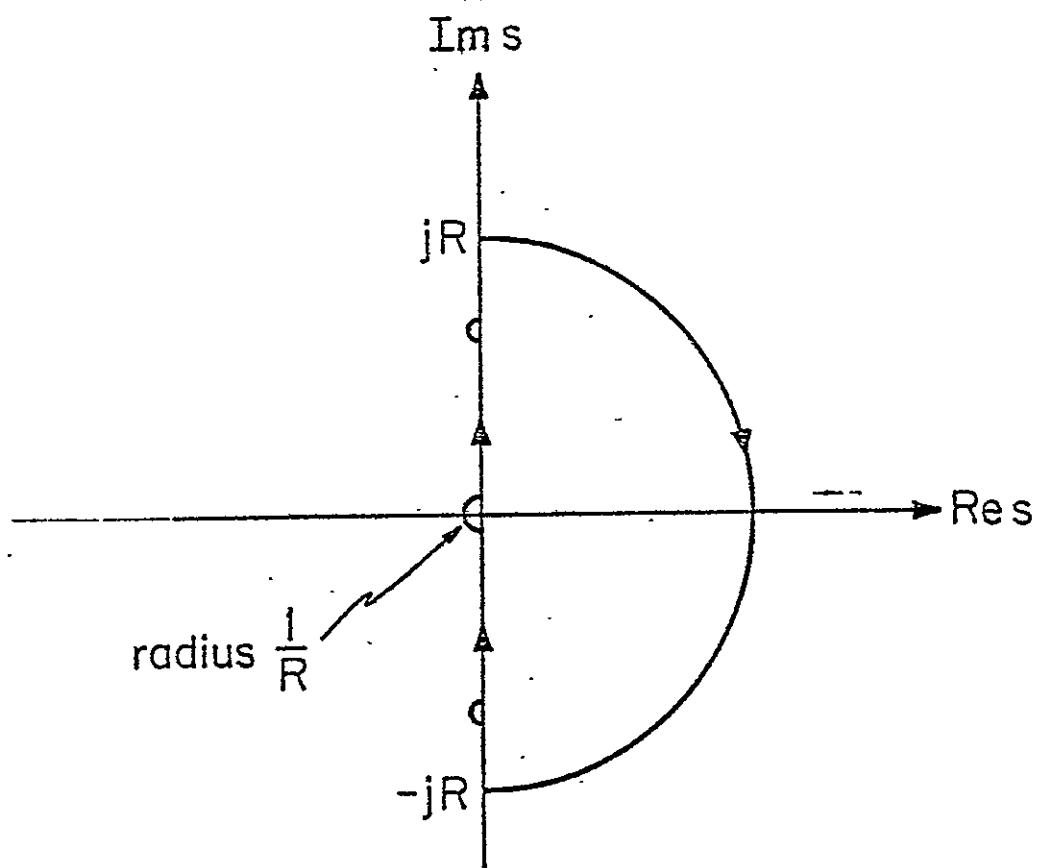


Fig. 2.2: Nyquist contour  $D_R$  which encloses all zeros of  $\phi_{0L}(s)$  in the CRHP, avoiding imaginary zeros by indentations of radius of  $1/R$ .

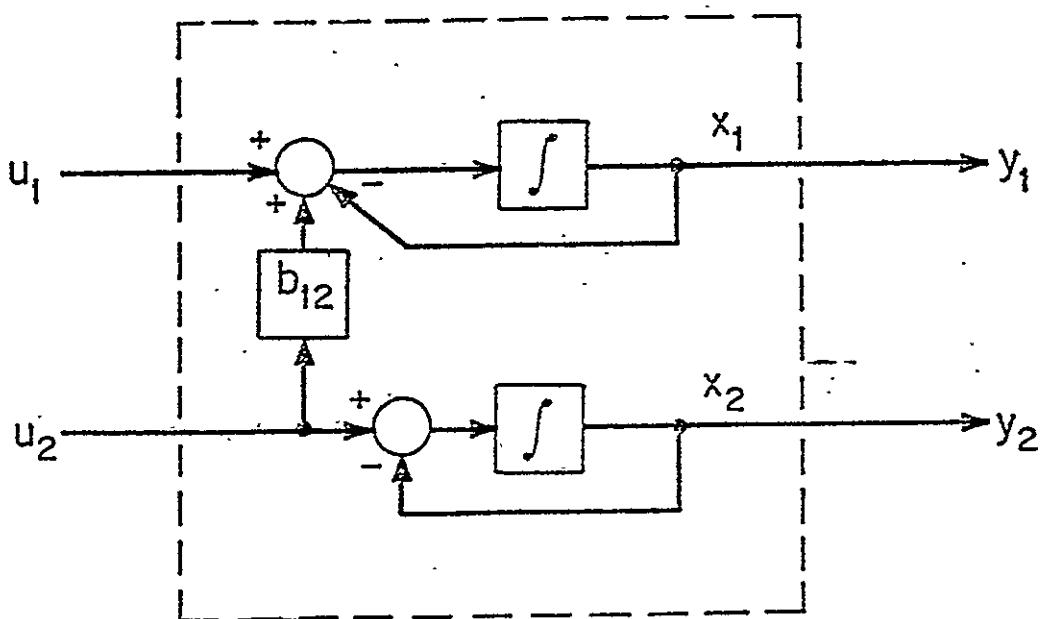


Fig. 2.3: Internal structure of Example 2.1

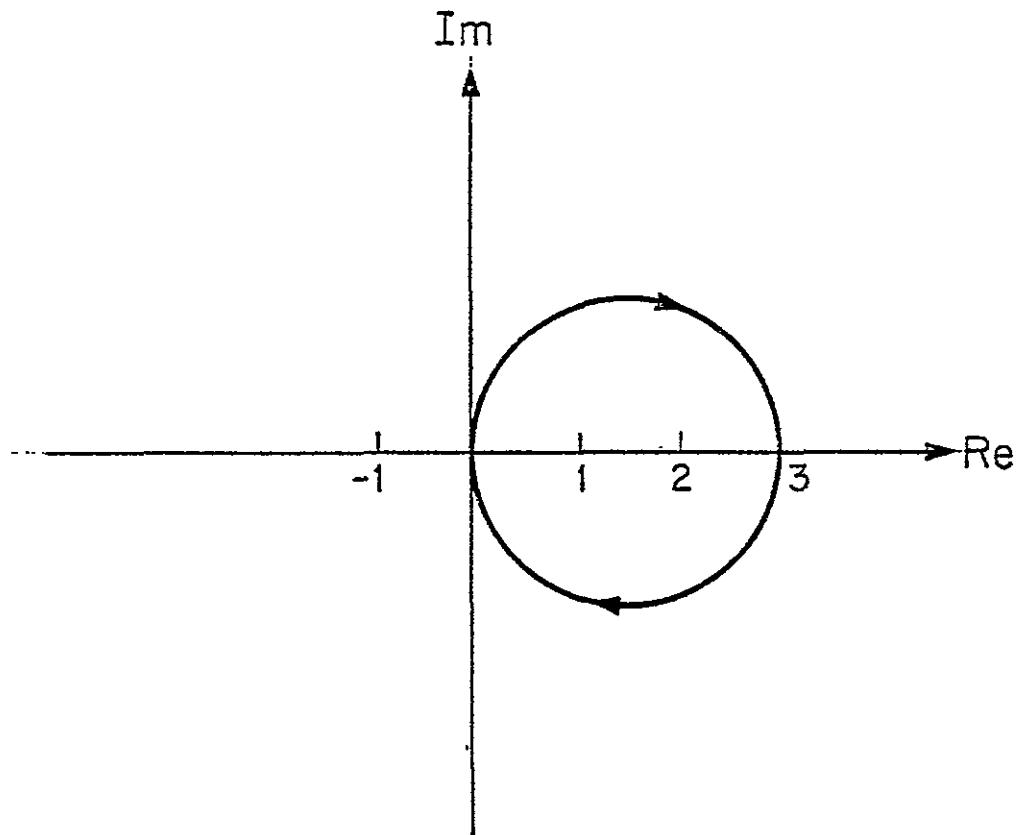


Fig. 2.4: Nyquist diagram of  $\frac{2s+3}{(s+1)^2}$ .

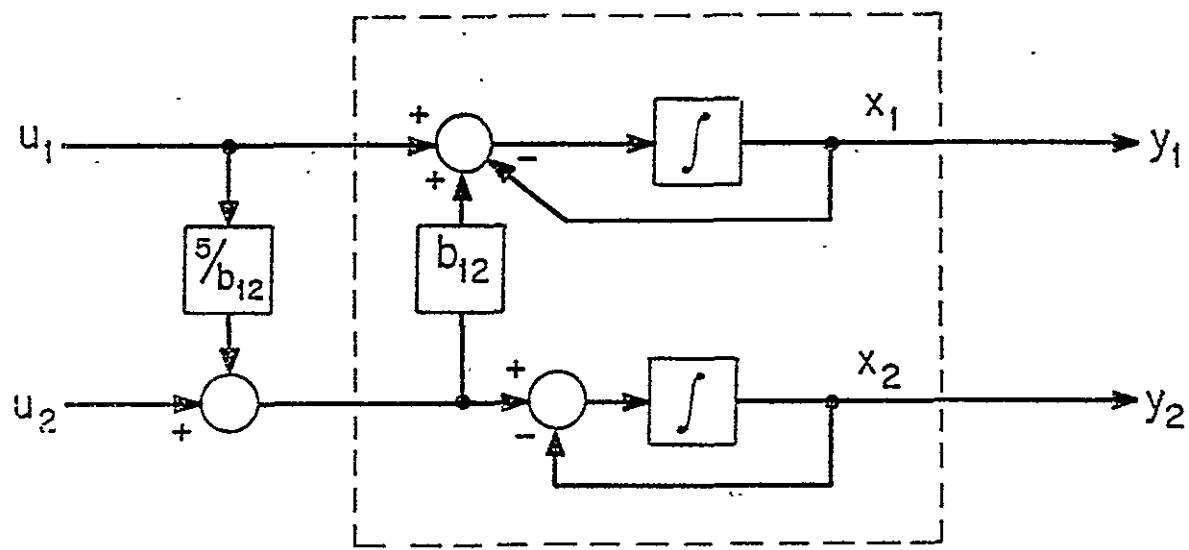


Fig. 2.5: Perturbation in open-loop system which makes closed-loop system unstable.

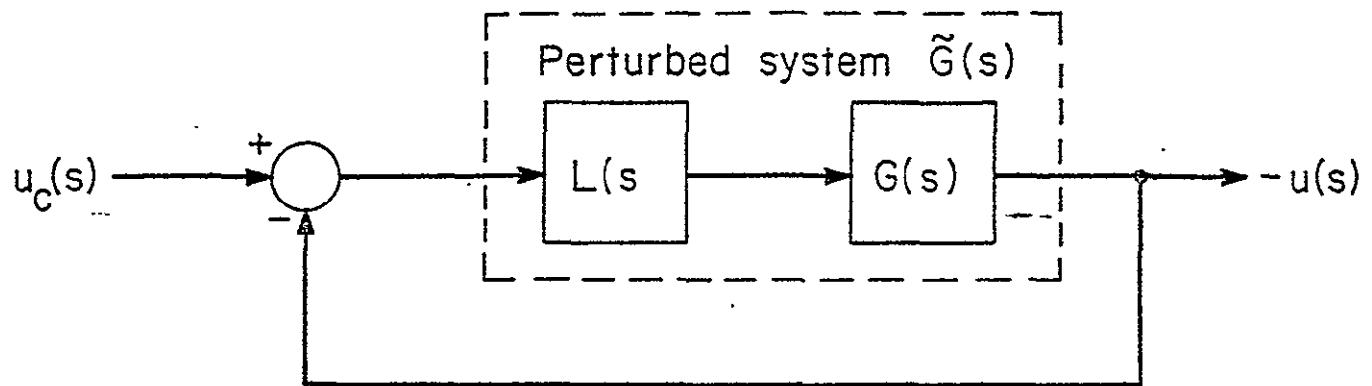


Fig. 2.6: Feedback system with multiplicative representation of uncertainty in  $G(s)$ .

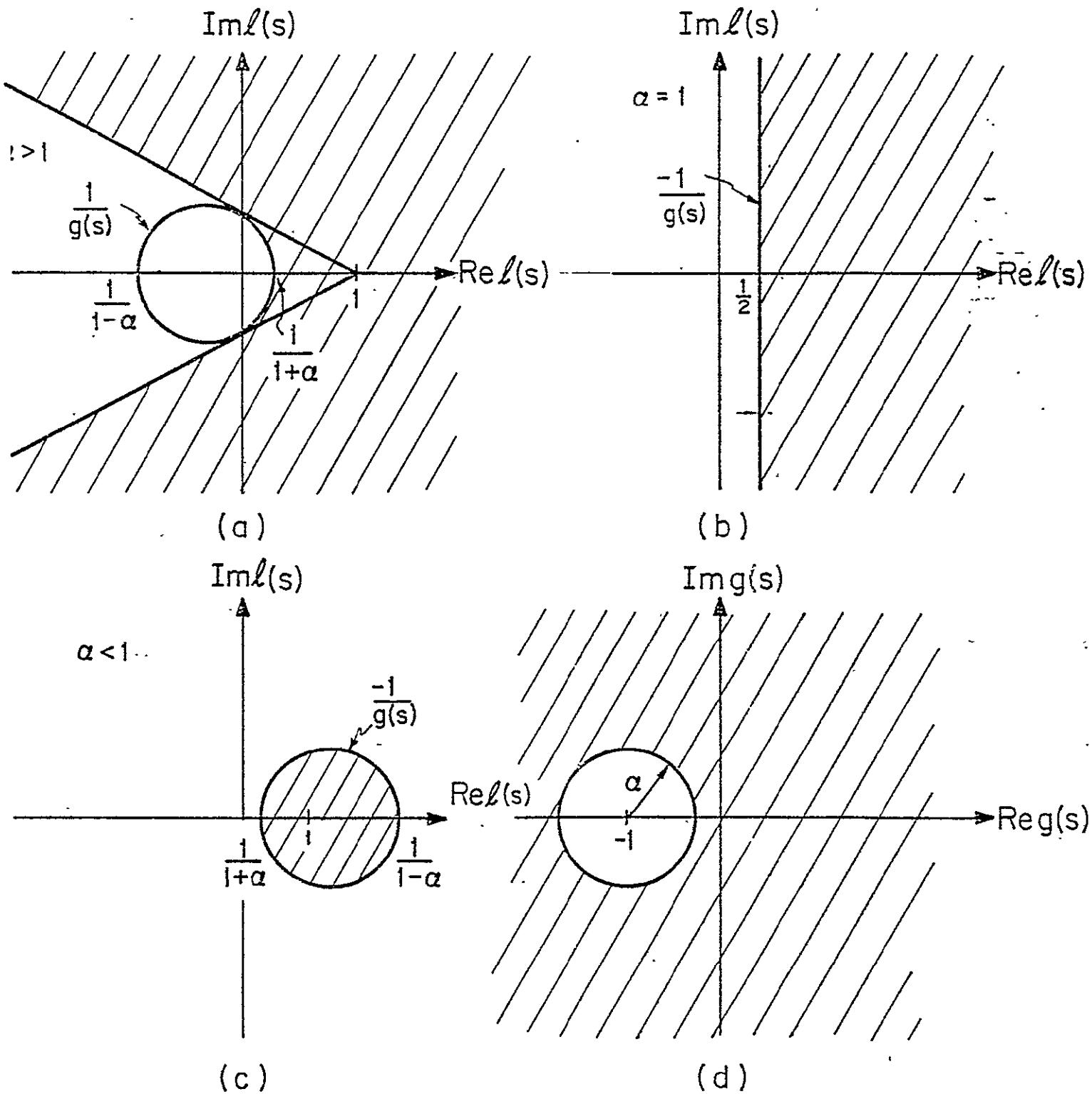


Fig. 2.7: Set of allowable values of  $l(s)$  and  $g(s)$  for various  $\alpha$  in Theorem 2.3.

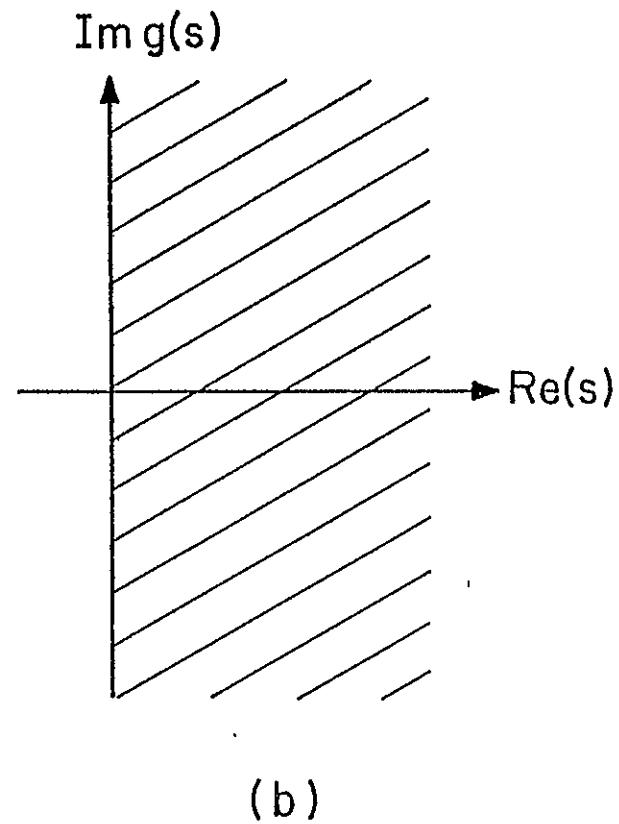
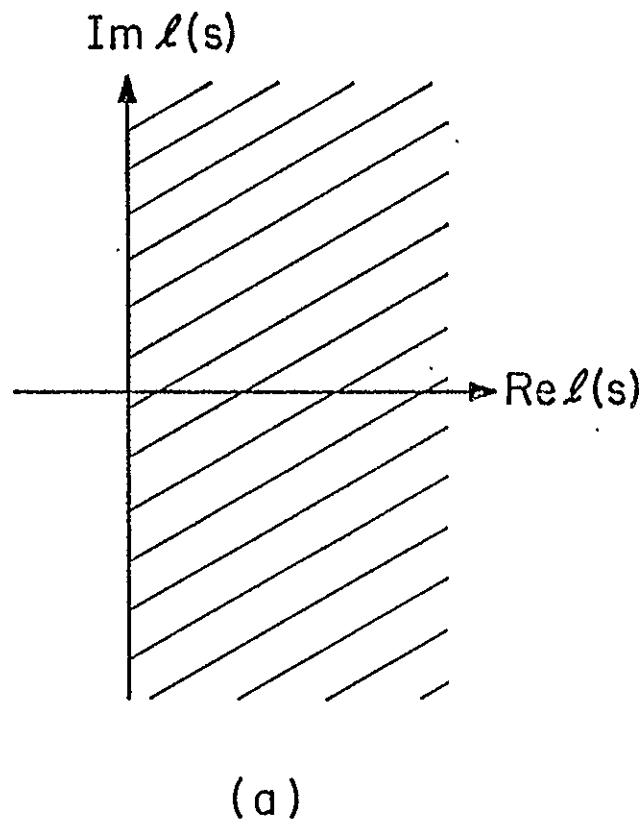


Fig. 2.8: Set of allowable values of  $\ell(s)$  and  $g(s)$  in Theorem 2.4.

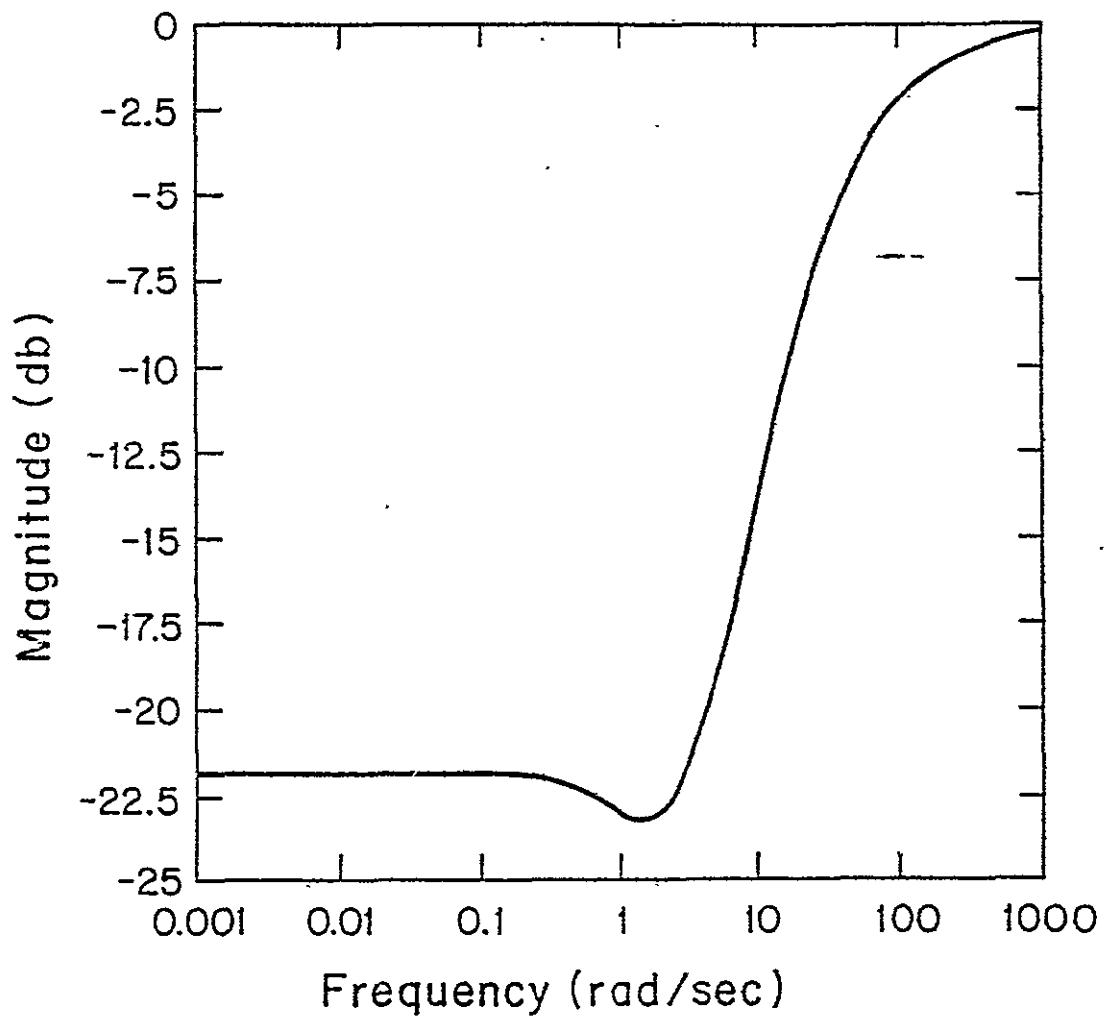


Fig. 2.9: Singular value plot of  $\sigma(I+G(j\omega))$  for Example 2.1 when  $\bar{b}_{12} = 50$ .

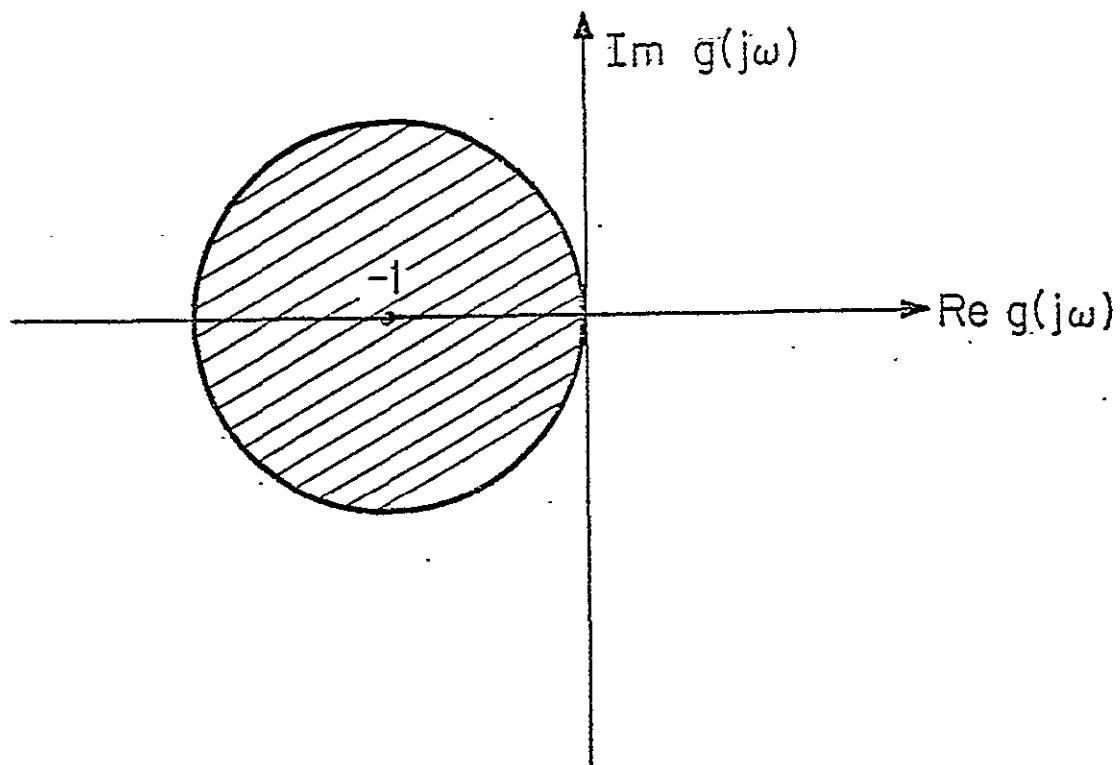


Fig. 3.1: Unit disk that Nyquist locus of  $g(j\omega)$  must avoid.

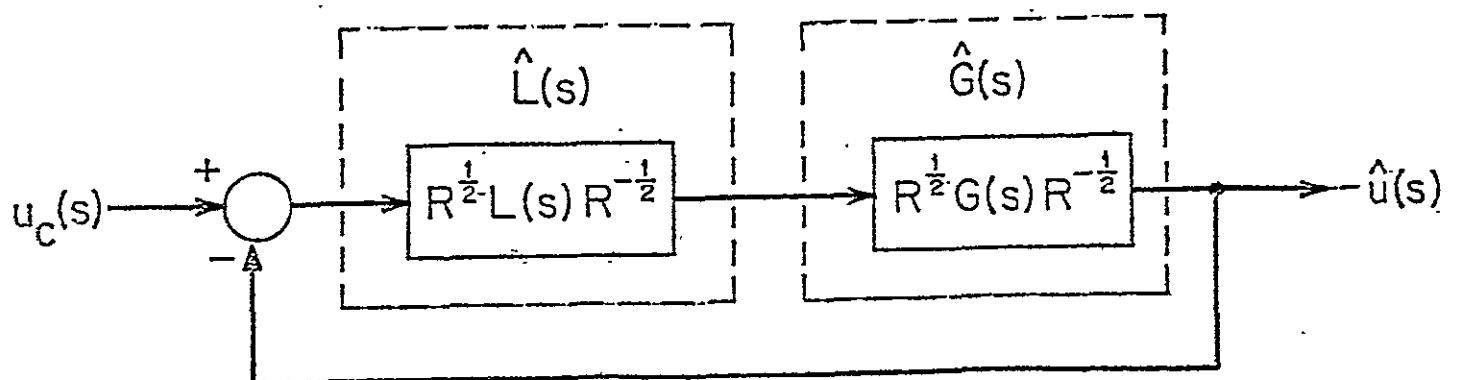


Fig. 3.2: Feedback system for stability margin derivation (compare Fig. 2.6).

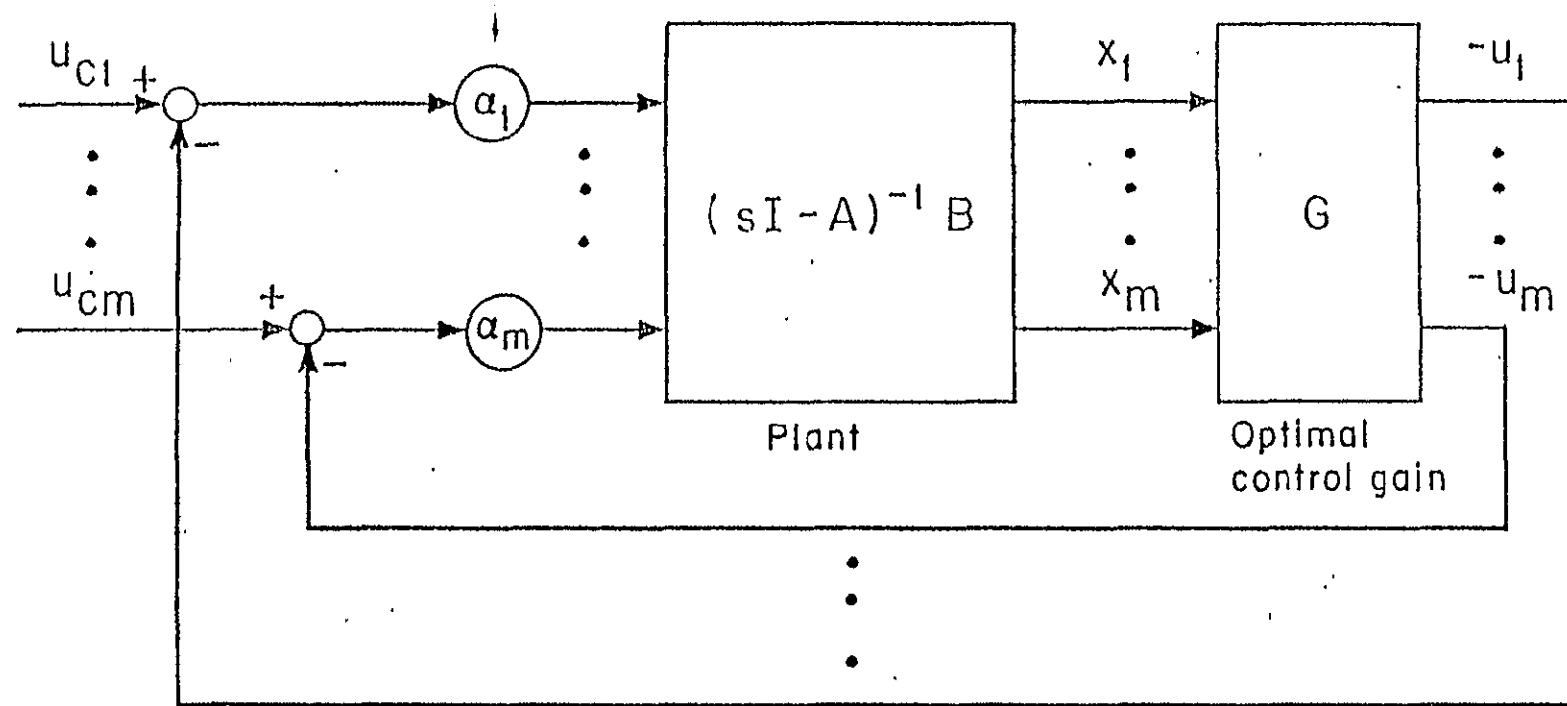


Fig. 3.3: Configuration for definition of multiloop LQ stability margins.

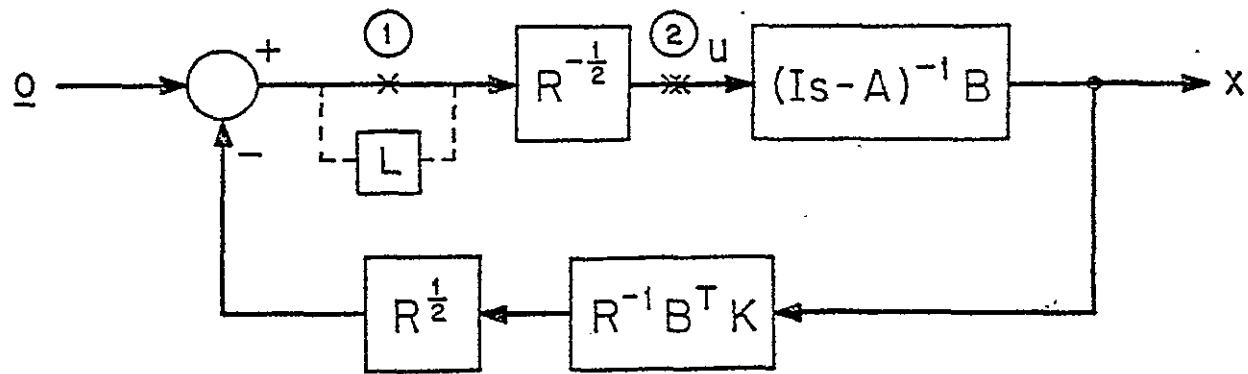


Fig. 3.4: LQ regulator with margins guaranteed at point ① for an  $R > 0$  and at both ① and ② for diagonal  $R > 0$ .

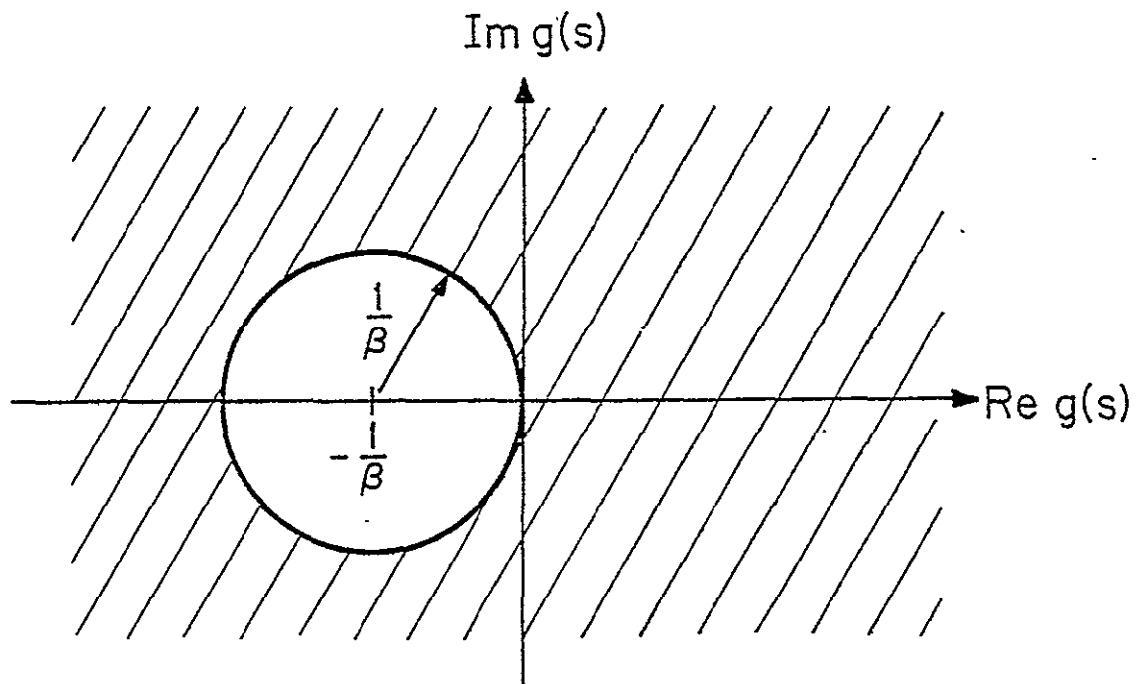


Fig. 3.5(a) : Set of allowable values of  $g(s)$  when  $|1+\beta g(s)| > 1$ .

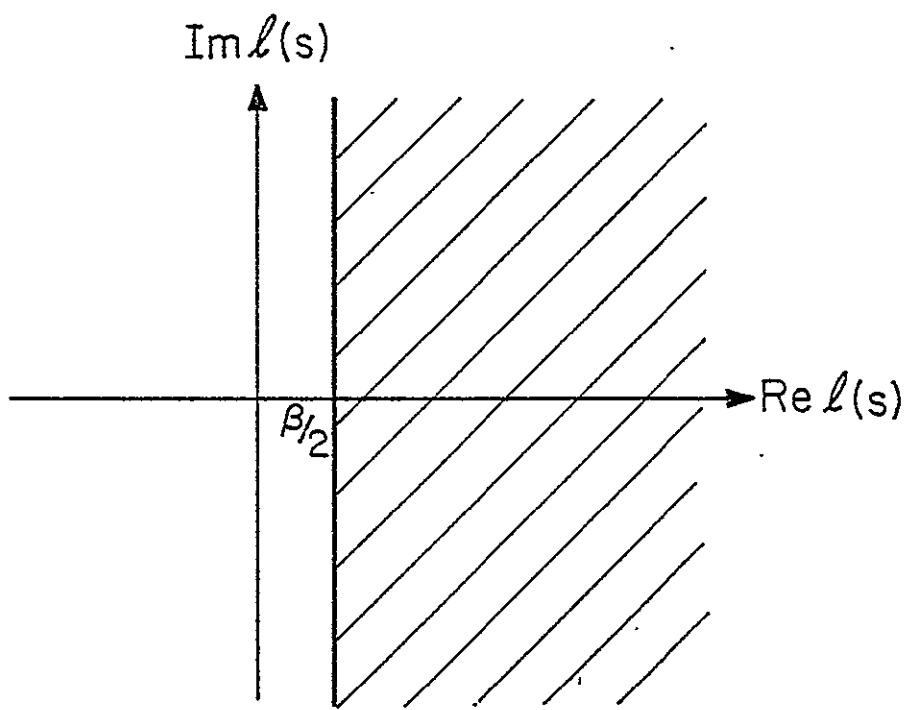


Fig. 3.5(b): Set of allowable values of  $\ell(s)$  when  $|\beta \ell^{-1}(s) - 1| < 1$  and  $0 < \beta < 2$ .

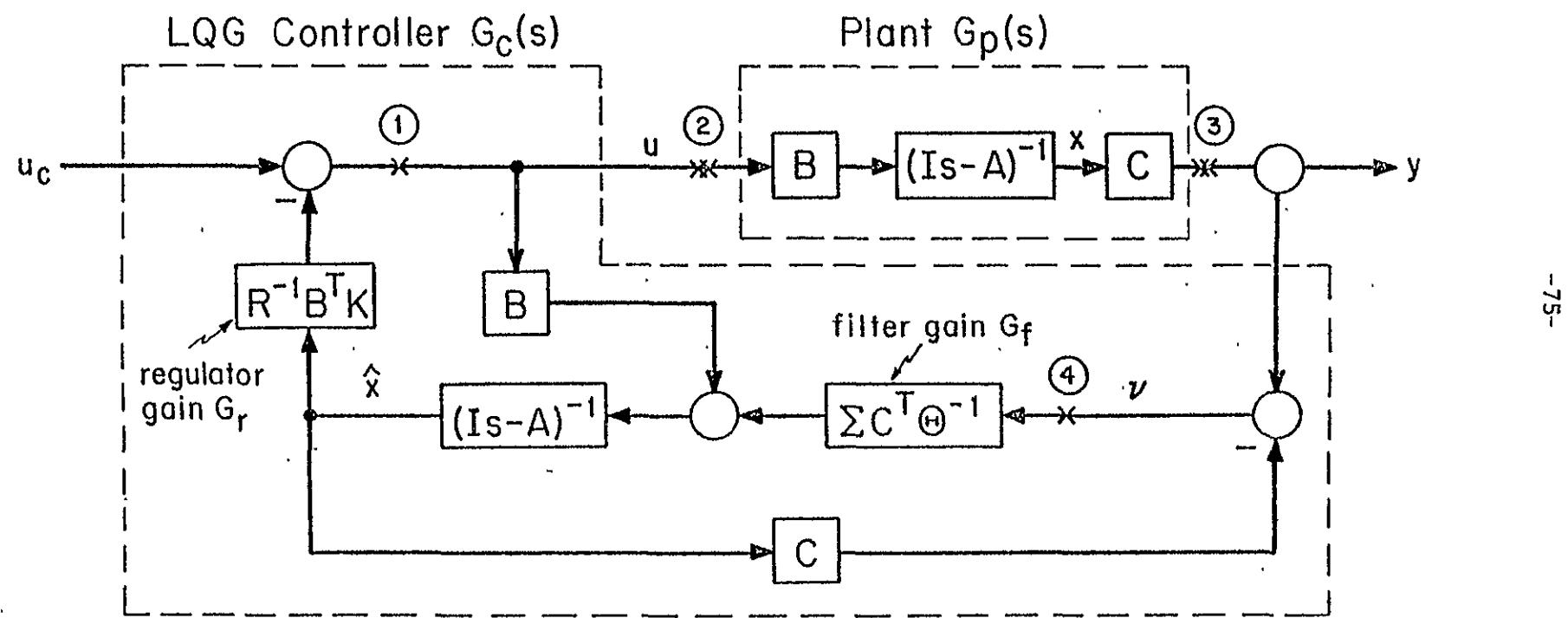


Fig. 3.6: LQG control system.

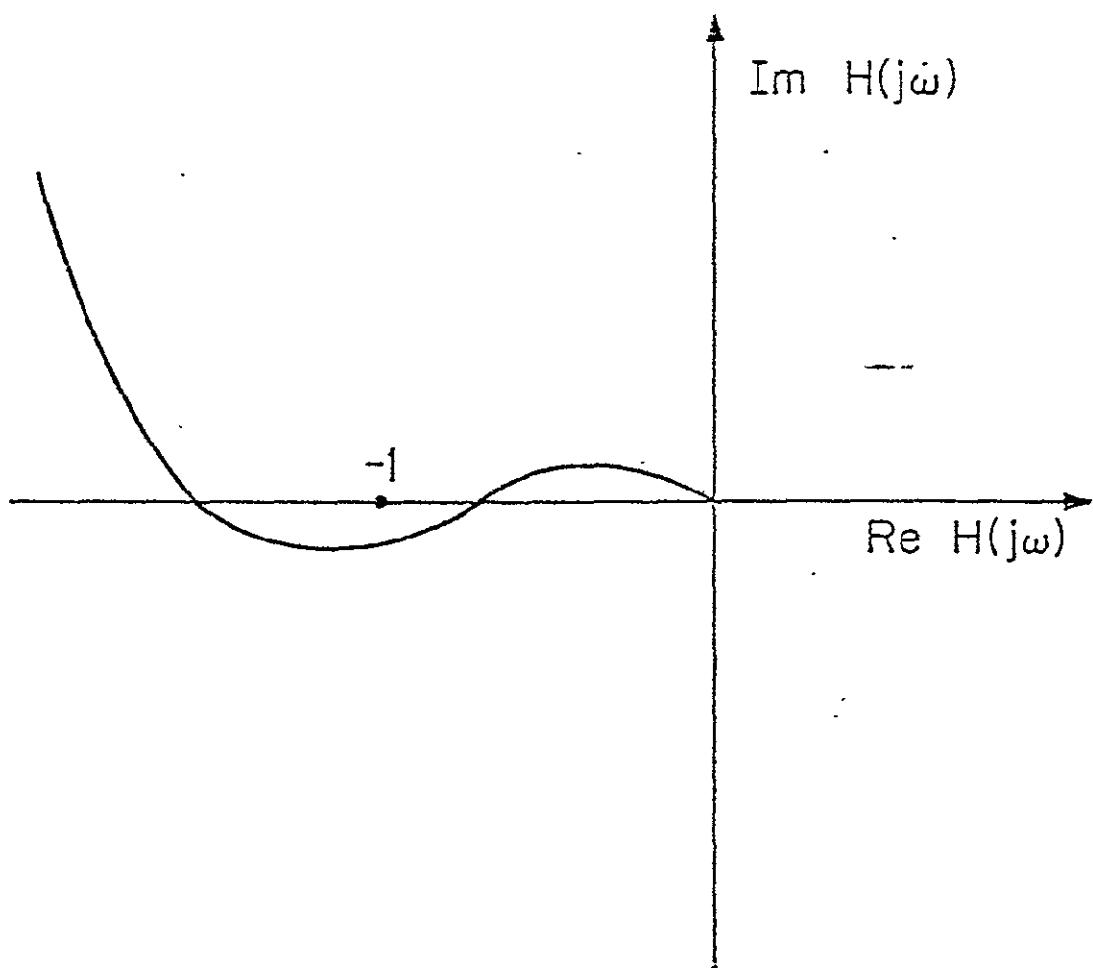


Fig. 3.7: Nyquist diagram for LQG design in [34].  
( $H(j\omega)$  = loop transfer function).